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MODELING OF BEHAVIOR AND INTELLIGENCE

BINARY RELATIONS IN THE THEORY OF ACTIVE SYSTEMS

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A base model of active systems is studied in close relation with binary relations. New analysis methods of optimal consistent planning are elaborated. A correspondence is shown to exist between binary relations and stimulation functions, between consistent plan sets and maximal sets, and between strongly consistent stimulation functions and strict ordering relations.

1. INTRODUCTION

The formulations, models, and methods of the theory of active systems, owing to its high level of development, are not only fruitful in control engineering, but are also effective in solving problems of operations research. This theory has been found to be closely related to many areas of operations research. The first result of such kind is the proof given in [1] to demonstrate the equivalence of optimal consistent planning and synthesis of optimal stimulation functions to the games Γ_1 and Γ_2, respectively, in the theory of games of nonantagonistic interests [2]. It is also related to the theory of collective selection [3, 4] and to the theory of multicriteria optimization [5, 6].

We study a base model of the theory of active systems in relation to the theory of binary relations [7, 8]. The theory of binary relations is a convenient tool for studying models of economic behavior of different types of agents and for generalizing the classical representation of such a behavior through utility functions.

The base model of a two-level active system with full information consists of a center and n active agents. For our purposes, it suffices to study a one-agent system. The state of the system is determined by the agent's state $y \in Y$, where Y is the set of possible states of the agent. The center assigns to the agent a desirable state $x \in Y$, while the agent chooses a state $y \in Y$ to suit his interests. The interests of the agent are described by a stimulation function $f(x, y)$; therefore, his decision making consists in solving the optimization problem

$$f(x, y) \to \max, \quad y \in Y.$$

In assigning a plan $x \in Y$ to the agent, the center must take account of its interest and the agent's behavior in the form of an objective function $\Phi(x, y)$. Moreover, it is imperative that the agent fulfill the plan assigned to him. This leads to an optimal consistent planning problem

$$\Phi(x, x) \to \max, \quad x \in S(f), \tag{1}$$

$$S(f) = \{x \in Y \mid f(x, y) \leqslant f(x, x)\, \forall\, y \in Y\}, \tag{2}$$

where $S(f)$ is a set of x-consistent plans which are profitable, by virtue of (2), for the agent to fulfill, i.e., the agent must choose a state $y \in Y$ equal to x.

A direct method of studying and generalizing the base model of the theory of active systems in terms of binary relations consists in introducing on the set Y a binary relation parametrically dependent on the plan x, i.e., in defining a subset of the three-dimensional Cartesian product $G \subset Y \times Y \times Y$,

$$y >_x z \Longleftrightarrow f(x, y) > f(x, z) \Longleftrightarrow (x, y, z) \in G. \tag{3}$$

Certain results in this direction are reported in [5, 6]. In [6], an active-system model with a multicriteria stimulation function for the agent is reduced to an active system with a scalar stimulation function for the agent by a suitable convolution.

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2. BINARY RELATIONS AND STRONGLY CONSISTENT STIMULATION FUNCTIONS

In order to apply numerical extremum search methods for solving problem (1), (2), the set of x-consistent plans must be represented in a suitable form (as the solution of a system of finite inequalities). Such a problem (construction of the set S) is investigated in [9, 10], which show that this problem is solvable for many classes of stimulation functions as well as for superpositions of simple stimulation functions.

In [1], the stimulation function is expressed through the functions of gain $h(y)$ and penalty $\chi(x, y)$ as

$$f(x, y) = h(y) - \chi(x, y),$$

where $\chi(x, x) = 0$. This paper also introduces an important class of strongly consistent stimulation functions with penalty functions satisfying the triangle inequality

$$\chi(x, y) \leqslant \chi(x, z) + \chi(z, y). \tag{4}$$

Omitting the details, we only mention that these stimulation functions guarantee the maximum operation efficiency of the x-consistent planning procedure (1), (2).

Introduction of binary relations on the set Y via (3) is rather unnatural, because it is cumbersome—not one relation, but a whole family of binary relations parametrized by the set Y need to be defined. The only advantage of such an introduction is that the binary relations $>_x$ are strict ordering relations.

Let us recall that a binary relation R defined on a set Y is a subset of the Cartesian product $Y \times Y$, $R \subset Y \times Y$. We write yRx if the ordered pair (y, x) belongs to R. The set-theoretic operations and relations $\cap$, $\cup$, $\subset$, and $\supset$ are introduced on a set of binary relations in an obvious manner. The binary relation $>$ is a strict partial ordering relation if it is transitive and antireflexive:

$$(y > x) \,\&\, (x > z) \Rightarrow y > z,$$
$$\forall\, y \in Y \ \neg\,(y > y).$$

The concepts of a plan and planning procedure are central in the theory of active systems. From this viewpoint, we can introduce a binary preference relation for the agent to prefer some realization from the admissible states set to the assigned plan.

The stimulation function $f(x, y)$ obviously generates an antireflexive binary relation R_f on the admissible state set Y by the rule

$$yR_f x \Longleftrightarrow f(x, y) > f(x, x). \tag{5}$$

Note that the binary relation (5), in general, is not a strict partial ordering relation. For an arbitrary antireflexive binary relation R on the set Y, there exists a stimulation function $f(x, y)$ that generates this relation by rule (5). The function

$$f_R(x, y) = \begin{cases} 1 & \text{for } \ yRx, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

is one such function. Hence there is a correspondence between stimulation functions and antireflexive binary relations. Furthermore, if a set of "maximal" agents

$$M(R) = \{x \in Y \mid \forall\, y \in Y \neg\,(yRx)\}$$

can be defined for a relation R, this set is the set of x-consistent plans for the corresponding stimulation function, i.e., we have the following set-theoretic equalities:

$$S(f) = M(R_f), \quad M(R) = S(f_R).$$

Among binary relations, an important class is represented by strict partial ordering relations. We have

THEOREM 1. (a) *The binary relation* $>_f$,

$$y >_f x \Longleftrightarrow f(x, y) > f(x, x),$$

generated by a strongly consistent stimulation function $f(x, y)$ *is a strict partial ordering relation.*

(b) *Conversely, for every strict partial ordering relation* $>$ *on a finite set* Y, *there exists a strongly consistent stimulation function generating this relation.*

For a strongly consistent stimulation function generating a strict partial order relation on an infinite set Y to exist, it is sufficient there exist a nonnegative bounded-above function $h(y)$ satisfying the monotonicity condition

$$y > x \Rightarrow h(y) > h(x).$$

For these conditions to hold, it is sufficient that the set Y be a compact and the relation $y > x$ be continuous [7].

In [7], the concept of the utility function—an indicator of the second kind—is generalized as a function $\varphi(x, y)$ on $Y \times Y$ whose sign is determined by the binary relation P:

$$xPy \Longleftrightarrow \varphi(x, y) > 0. \tag{7}$$

Relation (7) is similar to (5). Hence we have

Definition 1. *A stimulation function $f(x, y)$ is called the indicator function if $f(x, x) = 0$.*

An indicator function can be derived from any stimulation function $f(x, y)$ as follows:

$$f^{ind}(x, y) = f(x, y) - f(x, x). \tag{8}$$

It is a simple matter to verify that $S(f^{ind}) = S(f)$ and $R_{f^{ind}} = R_f$. If $f(x, y)$ is a strongly consistent stimulation function, its corresponding indicator function (8) satisfies the inverse triangle inequality

$$f^{ind}(x, y) \geqslant f^{ind}(x, z) + f^{ind}(z, y). \tag{9}$$

Therefore, an indicator function satisfying (9) is also said to be strongly consistent. We have

LEMMA 1. *For a function $f(x, y)$ to be a strongly consistent indicator function, it is necessary and sufficient that it admit the representation*

$$f(x, y) = \min_{i \in I} \left(f_i(y) - f_i(x)\right)$$

for some set of functions $f_i(x)$, $i \in I$.

COROLLARY 1 (modified Dushnik–Muller theorem). *Let $>$ be a strict partial ordering relation. Then there exists a set of functions $f_i(x)$, $i \in I$, such that*

$$y > x \Longleftrightarrow \forall i \in I : f_i(y) > f_i(x). \tag{10}$$

3. QUASICONSISTENT STIMULATION FUNCTIONS

To introduce strict partial ordering relations on a set Y, it suffices, as asserted by Theorem 1, to generate them by strongly consistent stimulation functions. But for their realization it suffices to deal with a quasiconsistent stimulation function defined by

Definition 2. *An indicator stimulation function $f(x, y)$ is said to be quasiconsistent if*

$$f(x, y) \geqslant \min\left(f(x, z), f(z, y)\right) \ \forall x, y, z \in Y.$$

Similar to Theorem 1, we have

THEOREM 2. *The binary relation $>_f$ on a set Y that generates a quasiconsistent stimulation function $f(x, y)$ is a strict partial ordering relation. Conversely, for every strict partial ordering relation $>$ on a set Y, there exists a quasiconsistent stimulation function $f(x, y)$ such that the relation $>$ is the same as the relation $>_f$.*

In light of Theorem 2, it is worthwhile to interpret the concept of quasiconsistency within the framework of the theory of active systems. Let us recall that the concept of strong consistency is closely related to the concept of optimality of the functioning mechanism. The quasiconsistency and strong consistency properties are such that one does not follow from the other. By way of example, let us consider the two stimulation functions defined on the set $Y = \{a, b, c\}$ below (Table 1).

The stimulation function $f_1(x, y)$ is quasiconsistent, but not strongly consistent. The stimulation function $f_2(x, y)$, on the contrary, is strongly consistent, but not quasiconsistent.

TABLE 1

$(f_1(x, y), f_2(x, y))$	a	b	c
a	$(0, 0)$	$(2, 0)$	$(3, 0)$
b	$(0, -2)$	$(0, 0)$	$(2, 0)$
c	$(0, -3)$	$(0, -2)$	$(0, 0)$

4. CONCLUSION

A relationship between the theory of binary relations and the base model of active systems is established. The theory of binary relations gives a new interpretation to optimal consistent planning and construction of a set of consistent plans $S(f)$, and paves the way for introducing the concept of an indicator stimulation function. From the standpoint of binary relations, the concept of a quasiconsistent stimulation function arising in the realization of strict partial ordering relations is natural. The concept of a strong consistent stimulation function is also interpreted in terms of strict partial ordering binary relations. This opened the way for studying the relationships between the theory of active systems and other areas in the theory of operations research.

I express my indebtedness to A. V. Malishevskii for the formulations of Lemma 1 and Corollary 1 as well as for valuable improvements in the paper.

APPENDIX

Proof of Theorem 1. (a) Let us assume that the antireflexive binary relation $>_f$ is not a strict partial ordering relation. Then $>_f$ is not transitive, i.e., there exist x', y', and $z' \in Y$ such that

$$x' >_f y', \quad y' >_f z', \quad \text{but} \quad \neg(x' >_f z'), \tag{11}$$

in contradiction to the equivalent system of inequalities

$$f(y', x') > f(y', y'), \quad f(z', y') > f(z', z'), \quad f(z', x') \leqslant f(z', z').$$

Adding these inequalities, we obtain

$$f(y', x') + f(z', y') > f(y', y') + f(z', x'), \quad \text{i.e.,}$$
$$h(x') - \chi(y', x') + h(y') - \chi(z', y') > h(y') + h(x') - \chi(z', x').$$

Hence, $\chi(y', x') + \chi(z', y') < \chi(z', x')$, contradicting the strong consistency of the stimulation function $f(x, y)$.

(b) For the strict partial ordering relation $y > x$, let us consider the stimulation function

$$f(x, y) = h(y) - \chi(x, y), \tag{12}$$

where $h(y)$ is a function of the height of the point y for this relation. It is the least upper bound of the length of the sequences in the set Y:

$$y_1 < y_2 < \ldots < y_k = y,$$

$\chi(x, y)$ is the penalty function defined by the formula

$$\chi(x, y) = \begin{cases} 0, & y = x, \\ (1 - \varepsilon)(h(y) - h(x)), & y > x, \\ (1 - \varepsilon)(h(y) - h(x)) + \varepsilon H, & \neg(y > x), \quad y \neq x, \end{cases} \tag{13}$$

$0 < \varepsilon < 1/H$, where H is the maximal height of the points in Y for the relation $y > x$.

Obviously, $\chi(x, y) > 0$ if $y > x$. We prove that

$$f(x, y) > f(x, x) \Longleftrightarrow y > x.$$

Let $y > x$. Then $h(y) > h(x)$. Therefore

$$f(x, y) = h(y) - (1 - \varepsilon)(h(y) - h(x)) = h(x) + \varepsilon(h(y) - h(x)) > h(x) = f(x, x).$$

Assuming the contrary, i.e., $f(x, y) > f(x, x)$, we find that

$$h(y) - \chi(x, y) > h(x), \quad \text{i.e.,}$$
$$\chi(x, y) < h(y) - h(x) \leqslant (h(y) - h(x)) + \varepsilon(H - h(y) + h(x)).$$

From (13), we obtain

$$\chi(x, y) = (1 - \varepsilon)(h(y) - h(x));$$

consequently, $y > x$. We now prove that the penalty function (13) satisfies the triangle inequality

$$\chi(x, y) \leqslant \chi(x, z) + \chi(z, y).$$

Assume the contrary, i.e., there exist x', y', and $z' \in Y$ such that

$$\chi(x', y') > \chi(x', z') + \chi(z', y').$$

If $y' > x'$, since $\chi(x, y) \geqslant (1 - \varepsilon)(h(y) - h(x))$, a contradiction is obtained from the inequalities

$$\chi(x', z') + \chi(z', y') \geqslant (1 - \varepsilon)(h(z') - h(x')) + (1 - \varepsilon)(h(y') - h(z'))$$
$$= (1 - \varepsilon)(h(y') - h(x')) = \chi(x', y').$$

If $(y' > x')$ is not true, then both $z' > x$ and $y' > z'$ cannot be true. Let, for example, $\neg (z' > x')$. Then

$$\chi(x', z') + \chi(z', y') = (1 - \varepsilon)(h(z') - h(x')) + \varepsilon H + \chi(z', y')$$
$$\geqslant (1 - \varepsilon)(h(z') - h(x')) + \varepsilon H + (1 - \varepsilon)(h(y') - h(z'))$$
$$= (1 - \varepsilon)(h(y') - h(x')) + \varepsilon H = \chi(x', y').$$

This contradiction shows that $\chi(x, y)$ obeys the triangle inequality. This completes the proof of the theorem.

Proof of Lemma 1. The function

$$f(x, y) = \min_{i \in I}(f_i(y) - f_i(x))$$

satisfies the inverse triangle inequality

$$f(x, y) \geqslant f(x, z) + f(z, y) \, \forall \, x, y, z \in Y. \tag{14}$$

Moreover, $f(x, x) = 0$. Therefore $f(x, y)$ is a strongly consistent stimulation indicator function. Conversely, if Y is a finite set, then for any function $f(x, y)$ satisfying the inverse triangle inequality and vanishing on the diagonal $f(x, x) = 0$, there exists a set $|Y|$ of scalar functions $g_1(x)$, $g_2(x), \ldots, g_{|Y|}(x)$ such that

$$f(x, y) = \min_{i=1,2,\ldots,|Y|}(g_i(y) - g_i(x)).$$

As $g_i(y)$ it suffices to take $f(i, y)$. By virtue of (14), we have $f(i, y) - f(i, x) \geqslant f(x, y)$. Therefore

$$f(x, y) \leqslant \min_{i \in Y}(f(i, y) - f(i, x)).$$

To prove the equality, it suffices to take $i = x$. Then

$$\min_{i \in Y}\big(f(i, y) - f(i, x)\big) \leqslant f(x, y) - f(x, x) = f(x, y).$$

Proof of Corollary 1. By Theorem 1, the relation $>$ is equivalent to the relation $>_f$ for some strongly consistent stimulation function $f(x, y)$. The strongly consistent stimulation indicator function $f^{ind}(x, y) = f(x, y) - f(x, x)$ corresponds to the function $f(x, y)$:

$$y > x \Longleftrightarrow f^{ind}(x, y) > f^{ind}(x, x) = 0.$$

Hence, by Lemma 1, we obtain (10). This completes the proof of the corollary.

Proof of Theorem 2. We carry out the proof in two stages.

1. Let $f(x, y)$ be a quasiconsistent stimulation function, and let $y >_f x$ and $z >_f y$. Consequently,

$$f(x, y) > f(x, x) = 0, \quad f(y, z) > f(y, y) = 0.$$

Hence $f(x, z) > \min(f(x, y)$ and $f(y, z)) > 0 = f(x, x)$, i.e., $z >_f x$. This proves the transitivity property. Therefore, $>_f$ is a strict partial ordering relation.

2. Let $y > x$ be a strict partial ordering relation. Consider the stimulation function

$$f(x, y) = \begin{cases} 1, & y > x, \\ 0, & \neg(y > x). \end{cases}$$

We now show that relation $>_f$ is the same as the relation $>$. If $y >_f x$, then $f(x, y) > f(x, x) = 0$, i.e., $f(x, y) = 1$, and, consequently, $y > x$. But if $y > x$, then $f(x, y) = 1 > f(x, x) = 0$, i.e., $y >_f x$. It now remains to show that $f(x, y)$ is a quasiconsistent stimulation function. Let us assume the contrary, i.e., there exist x, y, and $z \in Y$ such that

$$f(x, z) < \min\big(f(x, y), f(y, z)\big).$$

Since $f(\cdot, \cdot)$ takes only one of two values, we find that

$$f(x, z) = 0, \quad f(x, y) = 1, \quad f(y, z) = 1,$$

i.e., $y > x$, $z > y$, but $\neg(z > x)$. This contradiction concludes the proof of the theorem.

REFERENCES

1. V. N. Burkov and V. V. Kondrat'ev, *Organized Systems: Their Functioning Mechanisms* [in Russian], Nauka, Moscow (1981).
2. V. A. Gorelik and A. F. Kononenko, *Theoretical Game-like Decision Making Models for Ecological-Economic Systems* [in Russian], Radio i Svyaz', Moscow (1982).
3. V. N. Burkov, A. K. Enaleev, and D. A. Novikov, "Functioning mechanisms for socio-economic systems with information communication," *Avtomat. Telemekh.*, No. 3, 3–25 (1996).
4. V. N. Burkov and A. K. Enaleev, "Stimulation and decision-making in the active system theory," *Mat. Soc. Sci.*, No. 27, 271–291 (1994).
5. M. B. Aidarkhanov, M. Z. Arslanov, and A. B. Murzakhmetov, "Consistent optimization in active systems: multicriteria conditionality, fuzziness, and relationship with vector optimization," in: *Problems in Information and Control Sciences* [in Russian], Gylym, Almata (1995).
6. M. Z. Arslanov, "Multicriteria and consistency in active systems," *Avtomat. Telemekh.*, No. 2, 162–168 (1997).
7. D. B. Yudin, *Computer-Aided Decision-Making Methods* [in Russian], Nauka, Moscow (1989).
8. B. A. Berezovskii, V. I. Borzenko, and L. M. Kempner, *Binary Relations in Multicriteria Optimization* [in Russian], Nauka, Moscow (1981).
9. A. A. Ashimov, V. N. Burkov, B. A. Dzhaparov, and V. V. Kondrat'ev, *Consistent Control for Active Production Systems* [in Russian], Nauka, Moscow (1986).
10. A. A. Ashimov, M. Z. Arslanov, V. N. Burkov, and B. A. Dzhaparov, *Properties and Construction of Consistent Plans: Indeterminacy, Risk, and Dynamics in Organized Systems* [in Russian], Institute of Control Sciences, Moscow (1984).