Proceedings of 15th International Conference on Systems Science. Wroclaw, 2004. Volume 2. P. 295 – 306.

Key words – game theory, decision-making, reflexivity, hierarchy of beliefs

Dmitry NOVIKOV, Alexander CHKHARTISHVILI

MODELS OF REFLEXIVE DECISION-MAKING

The paper contains the survey of the game-theoretical models of reflexive decision-making. Most of equilibrium concepts, used in the game theory, require that the parameters of the game are common knowledge – all agents know it, all agents know that all agents know it and so on ad infinitum. In the general case the agents have different beliefs about beliefs of each other, thus an infinite (reflexive) belief structure appears. For this case the concept of informational equilibrium is fruitful. The paper is devoted to the formulation of the reflexive model, and contains conditions of the reflexive equilibrium existence and stability, solution of the reflexivity depth problem for some cases, and examples.

1. INTRODUCTION

Nowadays game-theoretical models are widely spread in the descriptions of social and economical systems (e.g. [1-3]). A great variety of social and economical relations generates the variety of games' models. The paper is devoted to the consideration of the informational aspects of decision-making in conflict situations and, particularly, the role of mutual beliefs of the agents.

It is well known [2, 3], that the *normal form game* Γ_0 is described by:

i) the cortege $\Gamma_0 = \{N, (X_i)_{i \in N}, (f_i(\cdot))_{i \in N}\}$, which includes the set of the players (agents) $N = \{1, 2, ..., n\}$, sets of their feasible actions $(X_i)_{i \in N}$ and goal functions $(f_i(\cdot))_{i \in N}$, f_i: $\prod_{j \in N} X_j \to \Re^1$, $i \in N$ (hereafter $\Re^1$ denotes the set of real numbers);

ii) the information, which agents posses at the moments of decisions making.

Traditionally, in non-cooperative game theory it is assumed that the agents choose their actions simultaneously and independently, and the information about the game Γ_0 is *common knowledge* [2-7] among the agents, i.e. each agent knows: the set of the players, all feasible sets and goal functions; he also knows that all other agents know it, they know, that he knows, etc. ad infinitum. Informally, all agents know the game they play.

To choose the action, which maximizes his goal function, the agent has to model (predict) actions of his opponents. The process of such modeling is usually referred to as the *reflexion* [4-6]. At this stage information plays an essential role [4-6].

Agent's thoughts about the choice of his action include *strategic reflexion* – what will be the choice of the opponents? Different approaches to the modeling of these thoughts lead to different concepts of the game equilibrium. This paper follows the concept of *Nash equilibrium*, which is defined as the game outcome, stable from the unilateral deviations of the agents. More formally, vector of agents' actions $(x_1^*,...,x_n^*)$ is a Nash equilibrium iff

$$\forall i \in N \quad x_i^* \in \operatorname{Arg}\max_{x_i \in X_i} f_i(x_1^*,...,x_{i-1}^*, x_i, x_{i+1}^*,...,x_n^*).$$

It is worth underlying that to find his Nash-stable action, the *i*-th agent has to know *parameters of the game* – all feasible sets and goal functions – and he has to be sure that all other agents possess the same information. But this is not enough – he has to be sure, that his opponents know that he knows parameters of the game, that they know that he knows it and so on ad infinitum. Thus, Nash equilibrium is essentially based on the fact that the parameters of the game are common knowledge.

There exist several models, where strategic reflexion is more complicated, than the reflexion in the normal form game Γ_0 (including strategic reflexion in bimatrix games, studied in [5]). Among them: hierarchical games [8], informational extensions of games [9, 10], concept of correlated equilibrium [3] and decisions in threats-counterthreats [11]. Nevertheless, parameters of the game in all these models are common knowledge.

Unlike shortly mentioned above models of strategic reflexion, not all game parameters, considered in the current paper, are common knowledge. For the description of the model let's suggest that agents' goal functions depend not only on their actions, but also on parameter $\theta \in \Omega$ ("*state of the nature*") which is not common knowledge, i.e. goal function of the *i*-th agent is expressed in the form $f_i(\theta, x_1,..., x_n)$, $i \in N$. Then strategic reflexion logically is preceded by *informational reflexion* – thoughts of the agent about what each agent knows (beliefs) about parameter θ, and about beliefs of other agents and so on. It leads to the concept of the informational structure – hierarchy of agent's beliefs, which reflects his knowledge of the unknown parameter, of the other agents' beliefs, beliefs about beliefs and so on.

Within the framework of the probabilistic knowledge (agents' beliefs include the following components: probability distribution on the set of states of nature; probability distribution on the set of states of nature and distributions on the set of states of nature, that characterize beliefs of other agents and so on.), universal beliefs space was constructed in [12]. In this case the game formally is reduced to some bayesian game [2-5] with Bayes-Nash equilibrium, introduced by J. Harsanyi [13].

The concept of bayesian game usually assumes that beliefs of the agents (a priori probability distribution of the states of nature) are common knowledge (the possibility of the rejection of this assumption is discussed in [14]). Meanwhile, the construction of the universal beliefs space, suggested in [12], is so cumbersome, that it does not allow to find the solution of any "universal" bayesian game in general case (see also [15]).

Therefore, it is rational to consider particular case of agents' beliefs – *point belief structure* (agents have certain beliefs – which are described by elements of the set Ω – about the value of the state of nature, about the beliefs (also concrete) of other agents, etc. [16]). It makes it possible to define finite complexity of the point belief structure (PBS), which, in turn, allows not only to define constructively the informational equilibrium as the concept of game solution, which "corresponds" to the PBS [16] and generalizes Nash equilibrium, but to explore its properties: existence [16, 17], stability [18] and others, and to elaborate a set of applied models [5, 6, 19].

2. INFORMATION STRUCTURE

Consider finite set $N = \{1, 2, \ldots, n\}$ of agents. Let there exists uncertain parameter $\theta \in \Omega$ (the set Ω is common knowledge). Informational structure of the *i*-th agent I_i includes the following elements. Firstly – the *i*-th agent's beliefs $\theta_i \in \Omega$ about the state of nature (first-order beliefs). Secondly – his beliefs $\theta_{ij} \in \Omega$ about beliefs of *j*-th agent, $j \in N$ (second-order beliefs). Thirdly – his beliefs $\theta_{ijk} \in \Omega$ about beliefs of j-th agent about beliefs of *k*-th agent, $j, k \in N$ (third-order beliefs). And so on. The result is the hierarchy of the *i*-th agent beliefs.

In other words, informational structure I_i of the *i*-th agent is defined by the set of all possible values $\theta_{ij_1 \ldots j_l} \in \Omega$, where l runs through the set of non-negative integer numbers, $j_1, \ldots, j_l \in N$.

Analogously, the *informational structure I* of the whole game is defined by the set of all possible values $\theta_{ij_1 \ldots j_l} \in \Omega$, where l runs through the set of nonnegative integer numbers, $j_1, \ldots, j_l \in N$. Let's stress, that the whole informational structure I is not "observed" by the agents – each of them knows only corresponding substructure.

Thus, an informational structure is an infinite *n*-tree, which nodes correspond to certain information of real or phantom (see below) agents.

Reflexive game is the game, described by the following cortege:

$$\Gamma_I = \{N, (X_i)_{i \in N}, f_i(\cdot)_{i \in N}, I\},$$

where N – set of agents, X_i – set of the *i*-th agent feasible actions, $f_i(\cdot)$: $\Omega \times X_1 \times \ldots \times X_n \to \Re^1$ – his goal function, $i \in N$, I – informational structure.

It is worth noting, that the term "reflexive game" was introduced by V. Lefebvre in 1965 (see the detailed description in [4]). But the mentioned above paper contains only qualitative discussions of the reflexivity role without any formal models.

Thus a reflexive game is the generalization of the normal form game Γ_0 and corresponds to the case, when the information of the agents is described by the hierarchy of beliefs (PBS) about the state of nature (all other parameters of the game are assumed to be common knowledge). "Classical" normal form game is a particular case of the reflexive game – when the state of nature is common knowledge, informational equilibrium turns into Nash equilibrium.

Introduce the following notation: Σ_+ – set of all possible sequences of indexes from the set N; $\Sigma = \Sigma_+ \cup \varnothing$; $|\sigma|$ – number of indexes in the sequence σ (equals zero for the empty sequence).

If θ_i is the *i*-th agent beliefs about the state of nature, and θ_{ii} – his beliefs about his own beliefs, it is naturally to put $\theta_{ii} = \theta_i$. In other words, the *i*-th agent is correctly informed about his own beliefs and considers other agents to be correctly informed about their own beliefs, etc. Formally, it means that the *axiom of self-information* is true: $\forall\ i \in N\ \ \forall\ \tau, \sigma \in \Sigma$ $\theta_{\tau i i \sigma} = \theta_{\tau i \sigma}$. This axiom, particularly, means, that, knowing θ_τ for all $\tau \in \Sigma_+$, such that $|\tau| = \gamma$, one can find θ_τ for all $\tau \in \Sigma_+$, such that $|\tau| < \gamma$.

Together with informational structures I_i, $i \in N$, one can consider informational structures I_{ij} (informational structure of j-th agent from the i-th agent point of view), I_{ijk} and so on. Identifying the informational structure with the agent, who is described by this structure, one may say that, besides *n real agents* (i-agents, $i \in N$) with informational structures I_i, *phantom agents* (τ-agents, $\tau \in \Sigma_+$, $|\tau| \geq 2$) with informational structures $I_\tau = \{\theta_{\tau\sigma}\}$, $\sigma \in \Sigma$, also take part in the game. Phantom agents exist in the mind of real agents and influence on their actions (see below).

Consider some properties of informational structures.

Informational structures I_λ and I_μ ($\lambda, \mu \in \Sigma_+$) are *identical* iff:

1. $\forall\ \sigma \in \Sigma\ \ \theta_{\lambda\sigma} = \theta_{\mu\sigma}$;

2. the last indexes in the sequences λ and μ coincide.

Denote the identity of informational structures $I_\lambda = I_\mu$. First condition in the definition of identity is transparent, while the second one requires some explanation. The matter is that below the action of τ-agent is obtained on the base of his informational structure I_τ and goal function f_i, which is defined just by the last index in the sequence τ. That is why, it is convenient to consider identity of informational structures, including the identity of goal functions.

Proposition 1. $I_\lambda = I_\mu \Leftrightarrow \forall\ \sigma \in \Sigma\ \ I_{\lambda\sigma} = I_{\mu\sigma}$.

Proofs of propositions 1-6 are given in [5, 16, 17].

Proposition 2. $\forall\ i \in N\ \ \forall\ \tau, \sigma \in \Sigma\ \ I_{\tau i i \sigma} = I_{\tau i \sigma}$.

Properties of informational structures may be reformulated from "objective" terms to *τ-subjective* terms: for example, informational structures I_λ and I_μ ($\lambda, \mu \in \Sigma_+$) are τ-subjective identical iff $I_{\tau\lambda} = I_{\tau\mu}$ ($\tau \in \Sigma_+$). Below, all definitions and propositions are formulated in τ-subjective terms, implying that if τ is empty sequence, then «τ-subjective» means «*objective*».

λ-agent is τ-subjectively *adequately informed* about μ-agent beliefs (shortly – about μ-agent) iff $I_{\tau\lambda\mu} = I_{\tau\mu}$ ($\lambda, \mu \in \Sigma_+$, $\tau \in \Sigma$). Denote this property: $I_\lambda >_\tau I_\mu$.

Proposition 3. Each real agent τ-subjectively regards himself adequately informed about any agent: $\forall\ i \in N\ \ \forall\ \tau \in \Sigma\ \ \forall\ \sigma \in \Sigma_+\ \ I_i >_{\tau i} I_\sigma$.

λ-agent and μ-agent are τ-subjective *mutually informed* iff $I_{\tau\lambda\mu} = I_{\tau\mu}$, $I_{\tau\mu\lambda} = I_{\tau\lambda}$ ($\lambda, \mu \in \Sigma_+$, $\tau \in \Sigma$). Denote this property: $I_\lambda >\!\!<_\tau I_\mu$.

λ-agent and μ-agent are τ-subjective *equally informed about σ-agent* iff $I_{\tau\lambda\sigma} = I_{\tau\mu\sigma}$ ($\sigma, \lambda, \mu \in \Sigma_+$, $\tau \in \Sigma$). Denote this property: $I_\lambda >_\sigma <_\tau I_\mu$.

λ-agent and μ-agent are τ-subjective *equally informed* $\forall\ i \in N\ \ I_{\tau\lambda i} = I_{\tau\mu i}$ ($\lambda, \mu \in \Sigma_+$, $\tau \in \Sigma$). Denote this property: $I_\lambda \sim_\tau I_\mu$.

Proposition 4. $I_\lambda \sim_\tau I_\mu \Leftrightarrow \forall\ \sigma \in \Sigma_+\ \ I_\lambda >_\sigma <_\tau I_\mu$.

Proposition 5. $\forall\ \tau \in \Sigma$ the following conditions are equivalent:

1) any two real agents are τ-subjectively mutually informed;

2) all real agents are τ-subjectively equally informed;

3) $\forall\ i \in N$ informational structure $I_{\sigma i}$ τ- subjectively depends only from i:

$\forall\ \tau \in \Sigma\ \ (\forall\ i, j \in N\ I_i >\!\!<_\tau I_j) \Leftrightarrow (I_1 \sim_\tau \ldots \sim_\tau I_n) \Leftrightarrow (\forall\ i \in N\ \forall\ \sigma \in \Sigma\ \ I_{\tau\sigma i} = I_{\tau i})$.

The notion of informational structures' identity allows to define such an important property as complexity. Notice, that, besides the informational structure I, there exists enumerable set of structures I_τ, $\tau \in \Sigma_+$, and identity relation generates classes of pairwise identical structures. The number of these classes may be referred to as the *complexity of the informational structure*.

Informational structure I has *finite complexity* $\nu = \nu(I)$ iff there exists finite set of pairwise non-identical structures $\{I_{\tau_1}, I_{\tau_2}, \ldots, I_{\tau_\nu}\}$, $\tau_l \in \Sigma_+$, $l \in \{1, \ldots, \nu\}$, such that for any structure I_σ, $\sigma \in \Sigma_+$, one can find identical structure I_{τ_l} from this set. If there is not such a finite set, then informational structure I has infinite complexity: $\nu(I) = \infty$. Let's call *finite* an informational structure with finite complexity, otherwise – *infinite*.

It is obvious, that minimal complexity of any informational structure equals the number n of real agents. Any set (finite or infinite) of pairwise non-identical structures I_τ, $\tau \in \Sigma_+$, such that any structure I_σ, $\sigma \in \Sigma_+$, is identical to one of them, will be called the *basis* of informational structure I.

If the informational structure has finite complexity, then there exists a minimal length of indexes sequence γ, that, knowing all the structures I_τ, $\tau \in \Sigma_+$, $|\tau| = \gamma$, one can reconstruct all the other structures. This length characterizes minimal reflexivity rank, which is necessary for the informational structure description.

Let's denote that the informational structure I, $\nu(I) < \infty$, has *finite depth* $\gamma = \gamma(I)$ iff

1. $\forall\ I_\sigma$, $\sigma \in \Sigma_+$, $\exists$ identical structure I_τ, $\tau \in \Sigma_+$, $|\tau| \leq \gamma$;

2. for any positive integer number ξ, $\xi < \gamma$, there exists a structure I_σ, $\sigma \in \Sigma_+$, which is not identical to any structure I_τ, $\tau \in \Sigma_+$, $|\tau| = \xi$.

If $\nu(I) = \infty$, then the depth is infinite: $\gamma(I) = \infty$.

Having the description of informational structure and its properties, can consider the process of interactive decision-making of real and phantom agents, what leads to the concept of informational equilibrium.

3. INFORMATIONAL EQUILIBRIUM

Given the informational structure I, the informational structure of any agent (real and phantom) may be obtained. If it is assumed that any rational agent tries to maximize his goal function, then the choice by τ-agent of his action x_τ is determined by his informational structure I_τ, therefore, knowing this structure, one is able to model his reasoning and find the chosen action. When choosing his action, the rational agent models actions of his opponents (implements reflexivity). Thus, to define the game solution concept, it is necessary to take into account actions of real agents as well as actions of phantom agents.

The set of actions x_τ^*, $\tau \in \Sigma_+$, is *informational equilibrium* iff:

1. informational structure I has finite complexity ν;

2. $\forall \lambda, \mu \in \Sigma_+\ \ I_\lambda = I_\mu \Rightarrow x_\lambda^* = x_\mu^*$;

3. $\forall\, i \in N, \forall\, \sigma \in \Sigma \quad x^*_{\sigma i} \in \operatorname{Arg}\max_{x_i \in X_i} f_i(\theta_{\sigma i}, x^*_{\sigma i 1}, \ldots, x^*_{\sigma i, i-1}, x_i, x^*_{\sigma i, i+1}, \ldots, x^*_{\sigma i, n})$. (1)

First condition means that only a finite number of agents (real and phantom) take part in the game. Second condition reflects the requirement that equally informed agents with the same goal functions ought to choose equal actions. And, finally, third condition corresponds to the rationality of the agents – any agents is trying to maximize his goal function (by the choice of his own action), substituting those actions of the opponents, which seem him rational in the framework of his beliefs.

Proposition 6. If the informational equilibrium x_τ^*, $\tau \in \Sigma_+$, exists, then it consists of not more than ν pairwise different actions, and the system (1) contains not more than ν pairwise different equations.

Thus, to find the informational equilibrium x_τ^*, $\tau \in \Sigma_+$, it is sufficient to write ν conditions (1) – for each of ν pairwise different values x_τ^*, corresponding to pairwise different informational structures I_τ.

If all the agents are equally informed, then the complexity of informational structure is minimal and equals the number of real agents. In this case (1) turns into the definition of Nash equilibrium.

Conditions of the informational equilibrium existence, formulated in terms of goal functions' and feasible sets' properties [17], are the same, as they are for Nash equilibrium [2, 3].

A convenient tool of informational equilibrium exploration is the *reflexive game graph* [5]. Vertexes of this graph correspond to real and phantom agents (i.e. the number of vertexes equals ν – the complexity of informational structure), arrows come to vertexes-agents from those vertexes-agents (number of incoming arrows is $n-1$), whose actions define the value of formers' goals functions.

Example 1 (Cournot oligopoly). Let there are three agents with goal functions: $f_i(\theta, x_1, x_2, x_3) = (\theta - x_1 - x_2 - x_3)x_i - \frac{x_i^2}{2}$, $x_i \geq 0$, $i \in N = \{1, 2, 3\}$; $\theta \in \Omega = \{1, 2\}$.

Variable x_i may be treated as the output of the i-th agent production. Then the first summand is income (production of price and output), the second is costs of production. Denote shortly the agent, who beliefs that $\theta = 1$, a pessimist, and the agent, who beliefs that $\theta = 2$ – an optimist.

Let first and second agents are optimists, the third one is pessimist, and all agents are equally informed. Then, according to proposition 5 $\forall\, \sigma \in \Sigma$ the following holds: $I_{\sigma 1} = I_1$, $I_{\sigma 2} = I_2$, $I_{\sigma 3} = I_3$ (and, in accordance with the second point of the informational equilibrium definition, the same is true for x_σ^*). One can see, that any informational structure is identical to one of the three structures: $\{I_1, I_2, I_3\}$, which form the basis. That is why the complexity of the considered structure is three, and its depth equals one. Corresponding reflexive game graph is give by figure 1.

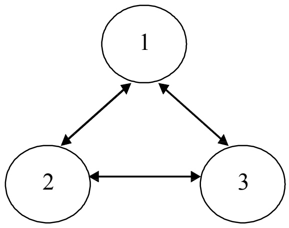

Fig. 1. Reflexive game graph in example 1 (all agents are equally informed)

Actions of real agents, which form informational equilibrium (correctly for this case – parametrical Nash equilibrium), are: $x_1^* = x_2^* = 1/2$, $x_3^* = 0$.

Consider another informational structure: let first two agents are optimists, the third one is pessimist, who is sure that all agents are equally informed pessimists; first two agents are equally informed and adequately informed about the third agent. Thus: $I_1 \sim I_2$, $I_1 > I_3$, $I_2 > I_3$, $I_1 \sim_3 I_2 \sim_3 I_3$. This conditions may be written (in accordance with propositions 1, 2, 5) as: $I_{12\sigma}=I_{2\sigma}$, $I_{13\sigma}=I_{3\sigma}$, $I_{21\sigma}=I_{1\sigma}$, $I_{23\sigma}=I_{3\sigma}$, $I_{3\sigma 1}=I_{31}$, $I_{3\sigma 2}=I_{32}$, $I_{3\sigma 3}=I_3$ (the same is true for x_σ^*).

Left-hand parts show, that any structure I_σ, $|\sigma|>2$ is identical to some structure I_τ, $|\tau|<|\sigma|$. Therefore depth of structure I is finite (and less than two). Right-hand parts show, that basis is formed by the following pair-wise different structures: $\{I_1, I_2, I_3, I_{31}, I_{32}\}$. That is why the complexity of the considered structure is five, and its depth equals two. Corresponding reflexive game graph is give by figure 2.

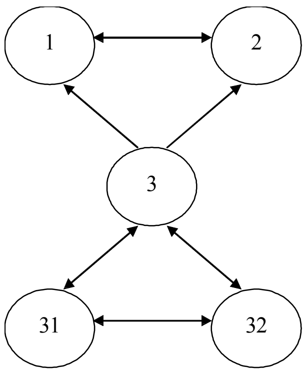

Fig. 2. Reflexive game graph in example 1
(first two agents are optimists, the third is pessimist, who is sure that all agents are equally informed pessimists; first two agents are equally informed and adequately informed about the third agent)

Actions of real agents, which form informational equilibrium, are: $x_1^* = x_2^* = 9/20$, $x_3^* = 1/5$. Thus, changing informational structure, one can change the choice of the agents.

4. REGULAR INFORMATIONAL STRUCTURES AND REFLEXIVE MAPPINGS

To consider regular informational structures [5], introduce recurrently the notion of the *regular finite tree* (RFT). If all of n agents are equally informed, then the informational structure has complexity n and depth one. Imagine this situation as the tree with the root and n leaves, connected by n edges. Further RFT may "grows" in the following way: for each leave τi, $\tau \in \Sigma$, $(n - 1)$ new edges are connected, thus $(n - 1)$ new leaves τij, $j = 1, \dots, i - 1, i + 1, \dots, n$ appear. Interpretation of the new RFT is the following: if there exists leave τi, $\tau \in \Sigma$, then τi-agent is equally informed with τ-agent (if τ is empty sequence, then τi-agent is real and his subjective beliefs coincide with objective beliefs).

Denote: the set of parametric (a parameter is vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in \Omega^n$) Nash equilibria

$$E_N(\boldsymbol{\theta}) = \{\{x_i\}_{i \in N} \in X' \mid \forall\, i \in N,\ \forall\, y_i \in X_i,\ f_i(\theta_i, x_1, \dots, x_n) \geq f_i(\theta_i, x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\}, \qquad (2)$$

$$E_N = \bigcup_{(\theta_1, \theta_2, \dots, \theta_n) \in \Omega^n} E_N(\theta_1, \theta_2, \dots, \theta_n).$$

Suppose, that on the lowest level $\{\theta_{\tau ij}\}_{j \in N}$ of the RFT subjective common knowledge of phantom agents takes place. Then from τi-agent point of view equilibria from the set $E_N(\{\theta_{\tau ij}\}_{j \in N})$ are possible. Introduce: the set of the i-th agent *best responses* on the choices of opponents from the set X_{-i} under the set Ω of possible states of nature:

$$BR_i(\Omega, X_{-i}) = \bigcup_{x_{-i} \in X_{-i},\, \theta \in \Omega} \operatorname{Arg}\max_{x_i \in X_i} f_i(\theta, x_i, x_{-i}),\ i \in N,$$

and the following sets: $E_N = \bigcup_{\boldsymbol{\theta} \in \Omega^n} E_N(\boldsymbol{\theta})$, $X_i^0 = Proj_i\, E_N$, $i \in N$, $X_{-i}^k = \prod_{j \neq i} X_j^k$, $i \in N$, $k = 0, 1, 2, \dots$, where

$$X_i^k = BR_i(\Omega, X_{-i}^{k-1}),\ k = 1, 2, \dots,\ i \in N. \qquad (3)$$

Depth of the agent's RFT equals his *reflexivity rank* plus one.

Mapping $BR_i(\cdot, \cdot)$: $\Omega \times X_{-i} \to X_i$ is called the *reflexive mapping* of the i-th agent, $i \in N$.

Proposition 7. $X_i^k \subseteq X_i^{k+1}$, $k = 0, 1, \dots, i \in N$, i.e. with the increase of the reflexivity rank sets of best responses (3) do not constrict.

Proofs of propositions 7 and 8 are given in [5].

Reflexive mapping of the i-th agent is *stationary* iff $X_i^k = X_i^{k+1}$, $k = 0, 1, \dots$.

Proposition 8. If the agents' reflexive mappings are stationary, then maximal rational rank of reflexivity equals two and the set of the i-th agent actions, which can be implemented (by varying his informational structure) as a components of informational equilibrium is X_i^0, $i \in N$, while the set of informational equilibria is $E = \prod_{i \in N} X_i^0$.

Thus, if the reflexive mappings of the agents are stationary, then the increase of the reflexivity rank over two is senseless.

5. STABLE INFORMATIONAL EQUILIBRIUM

One of the main features of Nash equilibrium is self-stability: if the normal form game is repeated several times and all the agents, except i, choose the same components of Nash equilibrium, then the i-th agent has no reason to deviate from the equilibrium. This fact is obviously related to the adequacy of agents' beliefs to reality.

In the case of informational equilibrium the situation may, generally, be different – if in one-shot game some agents (or even all agents) observe the result (choices of opponents, values of goal functions – see details below), which differ from the expected one. It may be caused as by incorrect beliefs about the state of nature, as by inadequate information about opponents' beliefs). In any case self-stability is destroyed – if the game is repeated again, actions of the agents may alter.

But in some cases self-stability may take place even under different (and, moreover, incorrect) beliefs of the agents. Informally, it is possible, when each agent (as real as phantom) observes the expected result of the game.

Introduce *functions of observation* $w_i(\cdot): \Omega \times X' \to W_i$, $i \in N$, – mappings of vector (θ, x) into the element w_i of the set W_i. This element w_i is what the i-th agent observes after the game is played. Assume that the observation functions are common knowledge among the agents.

If $w_i(\theta, x) = (\theta, x)$, i.e. $W_i = \Omega \times X'$, then the i-th agent observes as the state of nature, as the actions of all the agents. On the opposite, if W_i contains only one element, then i-th agent observes nothing.

Let the informational equilibrium x_τ, $\tau \in \Sigma_+$, of the reflexive game exists. Fix $i \in N$. Agent i expects to observe

$$w_i\ (\theta_i, x_{i1}, \ldots, x_{i,i-1}, x_i, x_{i,i+1}, \ldots, x_{in}). \qquad (4)$$

But indeed he observes

$$w_i\ (\theta, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n). \qquad (5)$$

Therefore, stability for the i-th agent means that (4) and (5) (which are elements of the set W_i) coincide.

Let (4) and (5) are equal, i.e. i-th agent has no reasons to doubt in his beliefs. Is this fact sufficient for him to choose next time the action x_i again? It is obvious, that the answer is negative, what is illustrated by the following example.

Example 2. Let the gains of the agents in the reflexive bimatrix game with $\Omega = \{1, 2\}$ are given by figure 3. Assume, that second agent beliefs $\theta = 2$ to be common knowledge, while the first agent knows the true state of nature $\theta = 1$ and is adequately informed about the second agent: $\theta = \theta_1 = 1$, $\theta_2 = \theta_{21} = 2$. Thus, the reflexive game graph is the following: $1 \leftarrow 2 \leftrightarrow 21$. Let each agent observe his own gain and this fact is common knowledge.

$$\begin{matrix} \theta = 1 & & \theta = 2 \\ \begin{pmatrix} (1,1) & (0,0) \\ (0,1) & (2,0) \end{pmatrix} & & \begin{pmatrix} (0,1) & (1,2) \\ (1,1) & (2,2) \end{pmatrix} \end{matrix}$$

Fig. 3. Gain matrixes in example 2

The informational equilibrium is: $x_1 = x_2 = x_{21} = 2$, i.e. the first and the second agents as well as 21-agent (first agent in the mind of the second agent) choose the second actions (second row and second column). But the real state of nature $\theta = 1$ becomes true for the second agent (when he receives 0 instead 2). That is why, next time he will choose $x_2 = 1$ and induce the first agent to change his action to $x_1 = 1$.

Thus, for the informational equilibrium to be stable, it is necessary that ij-th agent's observation will confirm his expectations, $i, j \in N$. He expected to observe

$$w_j\ (\theta_{ij}, x_{ij1}, \ldots, x_{ij,j-1}, x_{ij}, x_{ij,j+1}, \ldots, x_{ijn}). \qquad (6)$$

Really (i.e. i-subjectively, as ij-агент exists only in the mind of i-agent), he observes:

$$w_j\ (\theta_i,\ x_{i1},\ \ldots,\ x_{i,j\text{-}1},\ x_{ij},\ x_{i,j+1},\ \ldots,\ x_{in}). \qquad (7)$$

Such a reasoning leads to the following general definition of stability.

Given the informational structure I, informational equilibrium $x_{\tau i}$, $\tau i \in \Sigma_+$, is *stable* iff

$$\forall\ \tau i \in \Sigma_+\ w_i(\theta_{\tau i},\ x_{\tau i1}, \ldots, x_{\tau i,i\text{-}1}, x_{\tau i}, x_{\tau i,i+1}, \ldots, x_{\tau in}) = w_i(\theta_\tau,\ x_{\tau 1}, \ldots, x_{\tau,i\text{-}1}, x_{\tau i}, x_{\tau,i+1}, \ldots, x_{\tau n}). \quad (8)$$

Informational equilibrium, which is not stable, will be referred to as *unstable* (see example 2).

Proposition 9. If the informational structure I has complexity ν and informational equilibrium $x_{\tau i}$, $\tau i \in \Sigma_+$, exists, then (8) consists of not more than ν pairwise different conditions.

Proofs of propositions 9 and 10 are given in [18, 19].

6. TRUE AND FALSE EQUILIBRIA

Stable informational equilibria may be divided into two classes: true and false equilibria. First, consider the example.

Example 3. Let there are tree agents with goal functions $f_i(r_i, x_1, x_2, x_3) = x_i - \dfrac{x_i(x_1 + x_2 + x_3)}{r_i}$, $x_i \geq 0$, $i \in N = \{1, 2, 3\}$. Goal functions are common knowledge up to the *types* of the agents – parameters $r_i > 0$. Assume, that $r_2 = r_3 = r$, $r_{21} = r_{23} = r_{31} = r_{32} = c$, and the first agent is adequately informed about others, while the second and the third agents consider all agents as equally informed. Another fact is also common knowledge: each agent knows his type and observes the sum of opponents' actions. Graph of corresponding reflexive game is presented in figure 4.

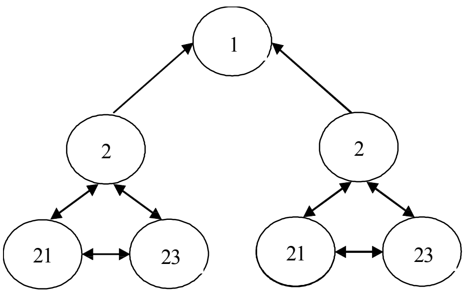

Fig. 2. Reflexive game graph in example 3

The unique informational equilibrium of the considered reflexive game is:

$$x_2 = x_3 = (3\ r - 2\ c)/4,\ x_{21} = x_{23} = x_{31} = x_{32} = (2\ c - r)/4,\ x_1 = (2\ r_1 - 3\ r + 2\ c)/4. \qquad (9)$$

Conditions of stability (8) take the form:

$$x_{21} + x_{23} = x_1 + x_3,\ \ x_{31} + x_{32} = x_1 + x_2. \qquad (10)$$

Notice, that (10) includes conditions only for 2- and 3-agents, as for 1-, 21-, 23-, 31-, 32-agents they are trivial. Substituting (9) in (10), obtain the necessary and sufficient condition of stability:

$$2\ c = r_1 + r. \qquad (11)$$

Let (11) holds. Then the actions of real agents are:

$$x_2 = x_3 = (3\,r - r_1) / 4,\ \ x_1 = (3\,r_1 - 2\,r\,) / 4. \qquad (12)$$

Suppose now, that types of agents became common knowledge. It is easy to test that (12) is the informational equilibrium in the case of common knowledge, as well.

Thus, under (11) a paradoxical situation occurs: beliefs of the second and of the third agents do not correspond to reality, while their actions (12) are the same, as they will be in the case of common knowledge. Such a stable informational equilibrium will be referred to as true equilibrium.

More correctly, let $x_{\tau i}$, $\tau i \in \Sigma_+$, is a stable informational equilibrium. This equilibrium is a *true equilibrium* iff $(x_1, \ldots, x_n)$ is an informational equilibrium under common knowledge about the state of nature θ (or about the types $(r_1, \ldots, r_n)$ of agents).

It follows from the definition above, that in the case of common knowledge any informational equilibrium (in particular – Nash equilibrium) is a true equilibrium. Consider one more case, when this fact takes place.

Proposition 10. Let the agents' goal functions are:

$$f_i\,(r_i, x_1, \ldots, x_n) = \varphi_i\,(r_i, x_i, w_i(x_{-i})),\ i \in N,$$

then any stable equilibrium is a true equilibrium.

Stable informational equilibrium, which is not true, will be referred to as the *false equilibrium*. Thus, a false equilibrium is such a stable informational equilibrium, which is not an equilibrium under common knowledge.

Example 4. Let the gains of the agents in the reflexive bimatrix game with $\Omega = \{1, 2\}$ are given by figure 5. Assume, that really $\theta = 2$, but all agents consider $\theta = 1$ to be common knowledge. Thus, the reflexive game graph is the following: $1 \leftrightarrow 2$. Let each agent observes (x_1, x_2) and this fact is common knowledge.

$$\theta = 1 \qquad\qquad \theta = 2$$
$$\begin{pmatrix} (2,2) & (4,1) \\ (1,4) & (3,3) \end{pmatrix} \qquad \begin{pmatrix} (2,2) & (0,3) \\ (3,0) & (1,1) \end{pmatrix}$$

Fig. 5. Gain matrixes in example 4

Choice of the first action by both agents is the informational equilibrium. If the real state of nature is common knowledge, then the choice of the second action by both agents will be the informational equilibrium. Thus, gains of agents are greater, than under the common knowledge.

7. CONCLUSION

Reflexive game is an efficient tool of modeling of interaction of agents, who make their decisions on the base of finite hierarchy of beliefs. If the dependence between the informational equilibrium and informational structure is known, then one can formulate and solve the problems of *informational management* – forming of informational structures, which lead to the required equilibrium. Nowadays theoretical results of the game-theoretical modeling of reflexive decision-making have practical applications in many fields of economics, psychology and management [5, 6, 19].

As the perspectives of further researches it is worth marking out exploration of dynamic (repeated and extensive form) and hierarchical reflexive games; exploration of interval, probabilistic and fuzzy finite informational structures and corresponding equilibrium concepts.

REFERENCES

[1] MAS-COLELL A., WHINSTON M.D., GREEN J.R., *Microeconomic theory*, N.Y., Oxford Univ. Press 1995.

[2] GOUBKO M., NOVIKOV D., *Game theory in organizational management*. Moscow, SINTEG 2002.

[3] MYERSON R.B., *Game theory: analysis of conflict,* London, Harvard Univ. Pr. 1991.

[4] LEFEBVRE V.A., *Conflicting structures*. Moscow, Radio and Svas 1967.

[5] NOVIKOV D., CHKHARTISHVILI A., *Reflexive games,* Moscow, SINTEG 2003.

[6] NOVIKOV D., CHKHARTISHVILI A., *Active forecast,* Moscow, ICS 2002.

[7] AUMANN R.J., HEIFETZ A., *Incomplete information*, Handbook of Game Theory, Vol. 3, Chapter 43, Amsterdam, Elseiver (forthcoming).

[8] GERMEIER Yu., *Games with non-antagonistic interests*, Moscow, Science 1976.

[9] KUKUSHKIN N., MOROZOV V., *Theory of non-antagonistic games*, Moscow, Moscow State University 1976.

[10] HOWARD N., *Theory of meta-games*, General systems, 1966, # 11, pp. 187 – 200.

[11] AUMANN R.J., MASHLER M., *The bargaining set for cooperative games*, Eds. M. Dresher, L.S. Shapley, and A.W. Tucker, Advances in Game Theory, Princeton, Princeton University Press 1964. pp. 443 – 447.

[12] MERTENS J.-F., ZAMIR S., *Formulation of Bayesian analysis for games with incomplete information*, Int. J. Game Theory, 1985, # 14, pp. 1– 29.

[13] HARSANYI J., *Games with incomplete information played by "Bayesian" players*, Management Science, Part I 1967, Vol. 14, # 3, pp. 159 – 182, Part II 1968, Vol. 14, # 5, pp. 320 – 334, Part III 1968, Vol. 14, # 7, pp. 486 – 502.

[14] SAKOVICS J., *Games of incomplete information without common knowledge priors*, Theory and Decision 2001, # 50, pp. 347 – 366.

[15] CHKHARTISHVILI A., *Bayes-Nash equilibrium: infinite-depth point belief structures*, Automation and Remote Control 2003, # 12, pp. 105 – 111.

[16] CHKHARTISHVILI A., *Informational equilibrium*, Large-scale systems control. Moscow, ICS 2003, Vol. 3, pp. 94 – 109.

[17] NOVIKOV D.A., CHKHARTISHVILI A.G., *Informational equilibrium: point belief structures*, Automation and Remote Control 2003, # 10, pp. 111 – 122.

[18] NOVIKOV D.A., CHKHARTISHVILI A.G., *Informational equilibrium stability in reflexive games*, Automation and Remote Control 2004 (forthcoming).

[19] NOVIKOV D.A., CHKHARTISHVILI A.G., *Applied models of informational management,* Moscow, ICS 2004.