REFLEXIVE MAPPINGS AND NONLINEAR DYNAMICS

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The paper considers reflexive mappings properties: it is proved that, when the agents in the framework of the game-theoretic model make their decisions on the base of the finite informational structures, actions, chosen by phantom agents, are defined by the system of nonlinear iterated mappings [1]. Exploration of the model allows concluding that the informational equilibrium is generally unstable under the increase of the reflexivity depth.

Consider the *informational structure* $I = (I_1, I_2, \ldots, I_n)$, where $I_i = (\theta_i, \theta_{ij}, \theta_{ijk}, \ldots)$, $i, j, k \in N = \{1, 2, .., n\}$, is the informational structure of i-th agent, $i \in N$, $\theta_i \in \Omega$ state of nature, $\theta_{ij} \in \Omega$ – his beliefs about the beliefs of j-th agent, $\theta_{ijk} \in \Omega$ – his beliefs about the beliefs of j-th agent about the beliefs of k-th agent and so on ad infinitum [2].

Reflexive game is the "normal form" game $\{N, (A_i)_{i \in N}, (f_i(\cdot))_{i \in N}, I\}$, where N is the set of players (agents), A_i is the set of i-th agent feasible actions, $f_i(\cdot)$: $\Omega \times A' \to \Re^1$ – his goal function, $A' = \prod_{i \in N} A_i$, $i \in N$, I – informational structure.

Denote Σ_+ – the set of all finite sequences of indexes from the set N, $\Sigma = \Sigma_+ \cup \varnothing$. $i\tau$-agent can be considered as the *phantom agent*, which exists in the mind of the real i-agent, $i \in N$, $\tau \in \Sigma$. The set of actions x_τ^*, $\tau \in \Sigma_+$, is the *informational equilibrium* [2], if the following conditions are satisfied:

1. the tree I contains a finite set of pairwise different sub-trees;

2. $\forall\, i \in N,\ \forall \lambda, \mu \in \Sigma_+\ \ \forall \lambda, \mu \in \Sigma\ \ I_{\lambda i} = I_{\mu i} \Rightarrow x_{\lambda i}{}^* = x_{\mu i}{}^*$;

3. $\forall\, i \in N,\ \forall\, \sigma \in \Sigma\ \ x^*_{\sigma i} \in \operatorname{Arg}\max\limits_{y_i \in A_i} f_i(\theta_{\sigma i}, x^*_{\sigma i1}, \ldots, x^*_{\sigma i,i-1}, y_i, x^*_{\sigma i,i+1}, \ldots, x^*_{\sigma in})$.

For regular informational structures [2] denote the set of parametric (vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in \Omega^n$ is a parameter) Nash equilibria

$$E_N(\boldsymbol{\theta}) = \{\{x_i\}_{i \in N} \in A' \mid \forall\, i \in N,\ \forall\, y_i \in A_i$$
$$f_i(\theta_i, x_1, \ldots, x_n) \geq f_i(\theta_i, x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\},$$

$$E_N = \bigcup_{(\theta_1, \theta_2, \ldots, \theta_n) \in \Omega^n} E_N(\theta_1, \theta_2, \ldots, \theta_n).$$

Suppose, that at the lowest level $\{\theta_{\tau ij}\}_{j \in N}$ of finite regular informational structure subjective common knowledge [2] of phantom agents takes place. From τi-th agents point of view possible outcomes of the game belong to the set $E_N(\{\theta_{\tau ij}\}_{j \in N})$ of equilibria. Introduce the set of best responces of i-th agent: $BR_i(\Omega, X_{-i}) = \bigcup\limits_{x_{-i} \in X_{-i},\, \theta \in \Omega} \operatorname{Arg}\max\limits_{x_i \in A_i} f_i(\theta, x_i, x_{-i})$, $i \in N$, and the following values and sets: $E_N = \bigcup\limits_{\boldsymbol{\theta} \in \Omega^n} E_N(\boldsymbol{\theta})$, $X_i^0 = Proj_i\, E_N$, $X_{-i}^k = \prod\limits_{j \neq i} X_i^k$, $i \in N$, $k = 0, 1, 2, \ldots$,

$X_i^k = BR_i(\Omega,\ X_{-i}^{k-1})$, $k = 1, 2, \ldots, i \in N$.

Mapping $BR_i(\cdot, \cdot): \Omega \times A_{-i} \to A_i$ is the *reflexive mapping* of i-th agent, $i \in N$ [2]. It was proved in [2], that $X_i^k \subseteq X_i^{k+1}$, $k = 0, 1, \ldots$, $i \in N$, i.e. with the increase of the *reflexivity rank k* the sets of best responses do not narrow. Reflexive mapping is *stationary* if $X_i^k = X_i^{k+1}$, $k = 0, 1, \ldots$. It was also proved in [2], that if the reflexive mappings are stationary, then the maximal rational rank of reflexivity equals 2, and the set of i-th agent actions, which can be implemented as the component of informational equilibrium, is X_i^0, $i \in N$. It leads to the following set of informational equilibria: $E = \prod_{i \in N} X_i^0$.

In many cases the increase of the reflexivity rank leads to the increase of uncertainty in the reflexive game outcome. It may be explained by the *nonlinearity* of the reflexive mappings. For example, in the reflexive model of Cournot oligopoly (with quadratic cost functions) best responses of agents are logistic mappings. Then the following asymptotically stable (and irrelevant to the initial stage) strategies of real agents are feasible: the choice of unique action, periodical behavior, chaotic or periodical behavior.

References

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