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FUZZY INCENTIVE PROBLEM

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Abstract: Game-theoretical model of the incentive mechanism is considered for the agency, which is embedded in fuzzy environment. The paper includes the solution of fuzzy incentive problem and the analysis of uncertainty influence on the efficiency of management.

Keywords: game theory, fuzzy models, decision making.

1. INTRODUCTION

Incentive problems for the organisations, which operate under uncertainty, are studied in numerous papers on the theory of active systems (Burkov and Enaleev, 1994; Burkov and Novikov, 1994; Burkov and Novikov, 1996), the theory of contracts (Grossman and Hart, 1983; Hart and Holmstrom, 1987; Hart, 1983) and other branches of management science. In accordance with the classification, introduced in (Novikov, 1997a), one should distinguish interval, stochastic and fuzzy uncertainty. In game-theoretical models under the interval uncertainty players posses information only about the set of feasible values of uncertain parameters, under the stochastic uncertainty the probability distribution is available, under the fuzzy uncertainty - the membership function. Till nowadays the least case has not attracted proper attention of operation researchers (exceptions are Novikov, 1997b; Novikov 1997c)), therefore this paper is devoted to the solution of the incentive problem for the agency under fuzzy external uncertainty, i.e. - uncertainty towards the state of nature. It is worth noting that the problem formulation is similar to the analogous problem of the contracts' theory (Hart and Holmstrom, 1987).

2. THE MODEL OF THE ORGANIZATION

Consider the organisation, which consists of the principal and the agent. Agent's strategy is the choice of action *y Î A*. The action, jointly with the state of nature *Q Î W*, leads to the output *z Î* A_0, which is determined by the "technological" function: *z = z(y, Q)*. Suppose that agent's goal function *f(z)* is:

$$f(z) = h(z(y,Q)) - c(z(y,Q)) \qquad (1)$$

the difference between his income *h(z)* and the penalties *c(z) Î M*, chosen by the principal. Principal's goal function coincides with his income *H(y)*, defined on the set of feasible actions. Note, that the theory of contracts manages with the inverse representation - "incentives minus costs". Under the assumptions, introduced below, both descriptions are equivalent. Moreover, the technic of the optimal penalty function design may be efficiently applied for the case when the penalties are added to principal's income (or the incentives are subtracted from his income).

The sequence of operation is the following: the principal reveals to the agent the penalty function, then the agent chooses his action, which is unobservable to the principal as well as the state of nature is unobservable until the strategies of both players are chosen, then the output is observed.

By contrast to the theory of contracts (when the probability distribution of the state of nature is considered to be common knowledge), it is assumed that both the principal and the agent have some fuzzy information about the state of nature (see the details below).

Introduce the following assumptions. Let SP' be the class of real value upper semi-continuous functions $q(x)$, defined on $\Re^1$, such that there exist $r^-, r^+ \in \Re^1$ (the possibility of $r^- = r^+ = r$, $r^- = -\infty$ or $r^+ = +\infty$ is not excluded) and $q(x)$ does not decrease if $x \le r^-$, is constant if $x \in [r^-, r^+]$ $(q(r^\pm) < +\infty)$ and does not increase if $x \ge r^+$. Functions, satisfying this conditions are referred to as quasi-singlepeaked.

A.1. $A_0 = A = \Re^1$.

A.2. $c(\cdot)$ - nonnegative uniformly upper-limited:

$$0 \le c(x, y) \le C < +\infty, \ \forall y \in A$$

piecewise-continuous function.

A.3. $h(\cdot), H(\cdot) \in SP'$.

To define the rational choice of the players, one should introduce the uncertainty remotion procedure. Suppose that the principal and the agent have the same fuzzy information $\underset{\sim}{P}(z, y)$: $A_0 \times A \to [0, 1]$ about the state of nature: $\underset{\sim}{P}(z, y)$ is the membership function of the output $z \in A_0$, which parametrically depends on agent's action $y \in A$ (if the fuzzy information $\underset{\sim}{P}_W(\theta)$ about $\theta \in \Omega$ is available, then $\underset{\sim}{P}(z, y)$ may be calculated from $\underset{\sim}{P}_W(\theta)$ and $z(y, \theta)$). As the output depends on the action and on the state of nature, then one have to obtain the fuzzy preference ordering (FPO) $m_{\underset{\sim}{R}_A}$ on the set of agent's actions. This FPO is induced by agent's goal function (1) and fuzzy information function $\underset{\sim}{P}(z, y)$. To solve this problem, the general approach, introduced in [10], may be applied. Finally, the FPO $m_{\underset{\sim}{R}_A}$ is defined in (Novikov, 1997b; Novikov, 1997c):

$$m_{\underset{\sim}{R}_A}(y_1, y_2) = \sup_{\substack{z,x\in A_0 \\ f(z)\ge f(x)}} \min\left[\underset{\sim}{P}(z, y_1),\ \underset{\sim}{P}(x, y_2)\right]. \qquad (2)$$

The fuzzy set of undominated actions has the following membership function (Orlovsky, 1981):

$$m^{ND}_{\underset{\sim}{R}_A}(x) = 1 - \sup_{y\in A}\left[m_{\underset{\sim}{R}_A}(y, x) - m_{\underset{\sim}{R}_A}(x, y)\right].$$

$$m^{ND}_{\underset{\sim}{R}_A}(x) = 1 - \sup_{y\in A}\left[\sup_{\substack{z,t\in A_0 \\ f(z)\ge f(t)}} \min\left\{\underset{\sim}{P}(z, y), \underset{\sim}{P}(t, x)\right\} - \right.$$
$$\left. - \sup_{\substack{z,t\in A_0 \\ f(t)\ge f(z)}} \min\left\{\underset{\sim}{P}(t, x), \underset{\sim}{P}(z, y)\right\}\right]. \qquad (3)$$

Rational behaviour of the agent implies the choice of the actions, which maximise (3), i.e. maximally undominated actions (MUA).

If preferences of the agent on the set of feasible outcomes depend on the penalty function, then the choice of the agent, generally, depends on this function too. Denote the unfuzzy set of MUA (it corresponds to the term "the set of implementable actions" in the theory of contracts) by $P(c)$:

$$P(c) = \left\{x \in A \,\middle|\, m^{ND}_{\underset{\sim}{R}_A}(x) = \max_{y\in A} m^{ND}_{\underset{\sim}{R}_A}(y)\right\}. \qquad (4)$$

The set of unfuzzy undominated actions $\left\{x \in A \,\middle|\, m^{ND}_{\underset{\sim}{R}_A}(x) = 1\right\}$ is referred to as the Orlovsky set (Orlovsky, 1981).

In the framework of the benevolence hypothesis (HB - the agent chooses from $P(c)$ the action, which is the most preferable from principal's point of view) the efficiency of management is defined as:

$$K(c) = \max_{y\in P(c)} H(y). \qquad (5)$$

The guaranteed efficiency is defined as the guaranteed value of principal's goal function over the set of MUA:

$$K^g(c) = \min_{y\in P(c)} H(y). \qquad (6)$$

The fuzzy set $\underset{\sim}{P}(z, y)$ is normal iff:

$$\forall y \in A \quad \sup_{z \in A_0} \underset{\sim}{P}(z,y) = 1 \,. \tag{7}$$

The fuzzy set $\underset{\sim}{P}(z, y)$ is α-normal iff:

$$\forall y \in A \quad \sup_{z \in A_0} \underset{\sim}{P}(z,y) = a \tag{8}$$

$$\forall z \in A_0 \; \exists y \in A: \quad \underset{\sim}{P}(z,y) = a \,. \tag{9}$$

A.4. $\underset{\sim}{P}(z, y)$ is α-normal.

Consider the following unfuzzy mathematical programming problem:

$$\begin{cases} f(z) \to max \\ \underset{\sim}{P}(z,y) \ge a \\ y \in A, z \in A_0 \end{cases} . \tag{10}$$

Lemma 1. Assume A.4 and FPO $m_{\underset{\sim}{R}_A}(x)$ is induced by agent's goal function (1) and fuzzy information function $\underset{\sim}{P}(z, y)$. If (z_0, y_0) is a solution of (10), then

$$m^{ND}_{\underset{\sim}{R}_A}(y_0) \ge a \,. \tag{11}$$

Let (z_0, y_0) be a solution of (10). It is sufficient to show that the following inequality takes place:

$$\sup_{y \in A}\left[\sup_{f(z) \ge f(t)} min\left\{\underset{\sim}{P}(z,y), \underset{\sim}{P}(t,y_0)\right\} - \right.$$
$$\left. - \sup_{f(t) \ge f(z)} min\left\{\underset{\sim}{P}(t,y_0), \underset{\sim}{P}(z,y)\right\}\right] \le 1 - a \,.$$

To the contrary, let there exist $\tilde{y}$ Î A and $e > 0$, such that

$$\sup_{f(z) \ge f(t)} min\left\{\underset{\sim}{P}(z,\tilde{y}), \underset{\sim}{P}(t,y_0)\right\} -$$
$$- \sup_{f(t) \ge f(z)} min\left\{\underset{\sim}{P}(t,y_0), \underset{\sim}{P}(z,\tilde{y})\right\} \ge 1 - a + e. \tag{12}$$

Choose $\tilde{z}$ Î A such that $\underset{\sim}{P}(\tilde{z}, \tilde{y}) \ge \alpha - \varepsilon$. As (z_0, y_0) is a solution of (10), then $f(z_0)$ ³ $f(\tilde{z})$ and $\underset{\sim}{P}(z_0, y_0) \ge \alpha$. Thus

$$\sup_{f(t) \ge f(z)} min\left\{\underset{\sim}{P}(t,y_0), \underset{\sim}{P}(z,\tilde{y})\right\} \ge$$
$$\ge min\left\{\underset{\sim}{P}(z_0,y_0), \underset{\sim}{P}(\tilde{z},\tilde{y})\right\} \ge a - e$$

which contradicts to (12). Q.E.D.

Thus, the problem of the analysis of α-undominated actions set is reduced to the exploration of standard unfuzzy mathematical programming problem (10). The following obvious lemma gives the set of sufficient conditions for the existence of the solution in a wide class of models.

Lemma 2. Assume that one of the following requirements are satisfied:

- A.4, A and A_0 are finite sets;
- A.4, A and A_0 are compact sets, f and $\underset{\sim}{P}$ are upper-semicontinous functions;
- A.4, A and A_0 are compact sets, f is upper-semicontinous function and $\underset{\sim}{P}$ is α-normal;

then the problem (10) has at least one solution.

The list of sufficient conditions, given by lemma 2, is far from being complete, but they correspond to most of the commonly used assumption and embrace many real incentive problems.

Corollary 3.
a) Under the conditions of lemma 1 and lemma 2 the set of α-undominated actions is not empty;
b) If A.4 is valid, $a = 1$, A and A_0 are compact sets and f and $\underset{\sim}{P}$ are upper-semicontinous functions, then the Orlovsky set is not empty and any solution of (10) belongs to this set.

The result of lemma 1 states that solutions of (10) are α-undominated actions of the agent. But, generally, some α-undominated action may not satisfy (10). The following trivial lemma defines the class of models, where such an opportunity is excluded.

Lemma 4. Assume A.4. Under the conditions of lemma 2 any α-undominated action belongs to the set of the corresponding unfuzzy mathematical programming problem solutions.

3. OPTIMAL INCENTIVE SCHEME

Lemma 5. Assume A.1 - A.3. Then the set of the agent's goal function maximums coincides with the closed interval $P = [z^-, z^+] \subseteq A_0$, where $z^- = \min \{z \in A_0 \mid h(z) \geq h(r) - C\}$,

$$z^+ = \max \{z \in A_0 \mid h(z) \geq h(r) - C\},$$

$$r \in [r^-, r^+] = \operatorname{Arg}\max_{y \in A} h(y). \quad (13)$$

Moreover, obviously, the set P is the set of implementable actions, while guaranteed implementation is valid for the following interval

$$P_d = [z^- + d,\ z^+ - d],\ d > 0. \quad (14)$$

Denote $y_1 = \arg\max_{y \in A} H(y)$ and suppose that $y_1 > r$ (the inversed inequality is analysed similarly). Consider the C-type incentive scheme (penalty function):

$$c_C(x, z) = \begin{cases} C, & z < x \\ 0, & z \leq x \end{cases}. \quad (15)$$

The action $x^* \in P$ gives maximal value to the agent's goal function. Fix some $x^* \in P$ and denote

$$Q(x^*, a) = \{y \in A \mid \underset{\sim}{P}(x^*, y) \geq a\}. \quad (16)$$

Lemma 6. Assume A.1-A.4. Then $\forall x \in P$ and $\forall y \in Q(x, a)$ there exists the penalty function - $c_c(x^*, z)$, such that the action y is α-undominated.

If the principal announces that he will use the C-type incentive $c_c(x^*, z)$, $x^* \in [z^-, z^+]$ then $x^* \in \operatorname{Arg}\max_{z \in A_0} f(z)$.

The conditions of the lemma imply that the corresponding set (16) is not empty. Thus $(x^*, y \in Q(x^*, a))$ is a solution of (10). Therefore, by lemma 1, $y \in Q(x^*, a)$ is an α-undominated action. Q.E.D.

If the rational behaviour of the agent is the choice of α-undominated actions, then the maximal set of implementable actions is given by the following condition:

$$S(a) = \bigcup_{x \in P} Q(x, a). \quad (17)$$

This set may be achieved by C-type penalty functions. Summarising this statement and results of the lemmas, the following theorem is valid.

Theorem 7. Assume A.1-A.4 and HB. Then:

a) for any feasible incentive scheme there exists the C-type penalty function, which has at least the same efficiency;
b) incentive problem's solution (the "jump" point x) coincides with a solution of the following problem: $\Phi(x) \to \max_{x \in S(a)}$, where the optimal value of x equals $x^* \in P$ and $S(a)$ is given by (16) - (17).

If the hypotheses of benevolence is not valid, then the efficiency of the C-type incentive scheme $c_c(x, x)$ is given by: $K(x) = K(c_c(x, x)) = \min_{y \in Q(x,a)} H(y)$. The optimal solution may be obtained from the coordinated (incentive compatible) planning problem: $K(x) \to \max_{x \in P}$.

4. UNCERTAINTY AND THE EFFICIENCY OF MANAGEMENT

Let us analyse the influence of uncertainty on the efficiency of management. Consider two fuzzy models, which differ only in information: $\underset{\sim}{P}_1(z, y)$ - in the first one and $\underset{\sim}{P}_2(z, y)$ - in the second.

In the first model the players posses more information than in the second model iff:

$$\forall y \in A,\ z \in A_0,\ \underset{\sim}{P}_1(z, y) \leq \underset{\sim}{P}_2(z, y). \quad (18)$$

Denote K_1 and K_2 - the appropriate efficiencies of management.

Theorem 8. Assume A.1-A.4. Then $K_1^g \geq K_2^g$, $K_1 \leq K_2$.

This result follows from the obvious conclusion: $\forall y \in A$ any fixed level sets of fuzzy information function $\underset{\sim}{P}_1(z, y)$ include corresponding sets of fuzzy information function $\underset{\sim}{P}_2(z, y)$ (see (16)).

The result of theorem 8 looks like not a trivial one. Under the hypotheses of benevolence the efficiency of management increases with the increase of uncertainty. For example, some unfuzzy undominated solution of fuzzy incentive problem may be more efficient, then the solution of the corresponding deterministic problem. The similar effects appear in the models under interval uncertainty (Novikov, 1997a).

This effect may be qualitatively explained in the following manner. The definition of agent's rational behaviour (the set of implementable actions) implies under the HB that he is indifferent between all the MUA. Thus, if there are several elements of this set, then the set of implementable actions includes corresponding deterministic one (both the principal and the agent adopt the widely used principle: "the less you know, the better you sleep"). If the HB is not valid, then the guaranteed efficiency of management satisfies the common sense - it decreases with the increase of uncertainty.

Consider the following example, which illustrates theoretical results.

Example. Let $A = A_0 = Â^1$; $h(z) = z - z^2/20$; $H(y) = y$; $\underset{\sim}{P}(z, y) = e^{-g(z-y)^2}$, where $g > 0$, $C = 1,8$. Then $P = [4; 16]$, $S(1) = P$.

Fix some a Î $(0; 1]$. The a-level sets of the fuzzy function $\underset{\sim}{P}(z, y)$ are: $[\, y - \frac{1}{g}\sqrt{ln(1/a)}\,;\, y + \frac{1}{g}\sqrt{ln(1/a)}\,]$.

Thus: $S(a) = [\, 4 - \frac{1}{g}\sqrt{ln(1/a)}\,;\, 16 + \frac{1}{g}\sqrt{ln(1/a)}\,]$.

With the decrease of a the set of implementable actions expands. With the increase of information (when g increases) narrows.

If the fuzzy information is given by

$$\underset{\sim}{P}(z, y) = \begin{cases} 1, & z \in [y-1, y+1] \\ 0, & z \notin [y-1, y+1] \end{cases}, \; y \text{ Î } A,$$

then the C-type incentive scheme $x^* = z^+ = 16$ under the HB induces the agent to choose the action $y_1^* = 17$. Consequently, the efficiency of management equals: $K_1 = H(y_1^*) = 17$. The efficiency of management in the corresponding deterministic model is $K_0 = H(x^*) = 16$. If the HB is not valid, then the choice of the agent is - $y_2^* = 15$, which leads to the decrease of the guaranteed efficiency of management: $K_2 = H(y_2^*) = 15$. Thus $K_2 < K_0 < K_1$.

5. CONCLUSION

The C-type incentive scheme is optimal in the fuzzy incentive problem under external uncertainty. The guaranteed efficiency of the fuzzy model management is less then the deterministic one and decreases with the increase of uncertainty.

The results of deterministic and stochastic models analysis may be efficiently applied (with slight modification) for fuzzy models. But in spite of the similarity with stochastic models, they differ essentially in the definitions of rational behaviour and optimal solutions. The class of fuzzy incentive problems seems to be rich enough both from theoretical and practical points of view, and requires the further exploration.

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