Algorithmic verification of feasibility for generalized median voter schemes on compact ranges

Korgin N.*

**Institute of Control Sciences of Russian Academy of Sciences, Moscow Russian Federation (Tel: +7495-335-60-37; e-mail: nkorgin@ipu.ru)*

Abstract: Barberá, Massó and Serizawa (1998) provided full characterization for class of strategy-proof social choice functions for societies where the set of alternatives is any full dimensional compact subset of a Euclidean space and all voters have generalized single-peaked preferences. They proved that this class is composed by generalized median voter schemes satisfying an additional condition, called the "intersection property". But according to their results in order to understand whether any generalized median voter scheme satisfies intersection property for given set of alternatives or not it was necessary to check all the alternatives from the set of unfeasible alternatives - addition of the set of feasible alternatives to minimal Cartesian product range, containing this set. So the number of alternatives to be checked, was infinite. In this paper it is proved, that it is enough to check finite number of alternatives from the set of unfeasible alternatives and constructive algorithm to determine alternatives that should be checked is provided.

1. INTRODUCTION

For societies with n agents facing a set Z of alternatives, a social choice function determines what alternative to choose for each possible profile of preferences. One of the important properties of social choice functions is strategy-proofness – when the best strategy for each agent is to report its preferences truthfully. But, in general, this property is hard to obtain – due to the Gibbard–Satterthwaite Theorem [see Gibbard (1973) and Satterthwaite (1975)] all social choice functions whose range contains more than two alternatives are either dictatorial or manipulable if all possible preferences over alternatives are admissible for all agents. But, applying some restrictions for the domain of admissible preferences, one can achieve existence of nondictatorial strategy-proof social choice functions. In this paper the setting is considered when the set of feasible alternatives Z is a full dimensional compact set in m-dimensional Euclidean space $\mathbb{R}^m$ and agent's preferences - multidimensional single-peaked with the added requirement that the unconstrained maximal element of these preferences (agent's top) belongs to Z.

Due to Barberá, Massó and Serizawa (1998), any social choice function in this setting is strategy-proof if and only if it is a generalized median voter scheme (GMVS) satisfying the intersection property for Z. This result, initially proved by Barberá, Massó and Neme (1997) for finite sets of alternatives follows those, obtained by Border and Jordan (1983) for whole $\mathbb{R}^m$ as the set of alternatives and Moulin's (1980) initial analysis of the one-dimensional case.

Structure of GMVS leads to the fact, that the result in each dimension is determined independently. Thus there is exist possibility, that the final outcome will be outside Z, but in the minimal Cartesian product range $\hat{B}(Z)$ containing this set even if tops of all agents belong to Z. In other words, an unfeasible alternative can be chosen as result of generalized median voter scheme application.

Intersection property was offered as a tool for checking, whether the GMVS respects feasibility for the set of feasible alternatives or not. But in order to make sure that any GMVS in hand satisfies intersection property for any set of alternatives Z it turns out, that all the unfeasible alternatives from set $\hat{B}(Z) \setminus Z$ must be inspected. This is quite inconvenient, when one works with continuous setting because there is infinite number of unfeasible alternatives to be checked. According to Bossert and Weymark (2006) and Barberá (2010), this results can be treated as some kind of frontline in solution of the problem considered. For finite setting of alternatives Nehring and Puppe (2007) provided alternative definition of intersection property that is more simple that original one, but still all the unfeasible alternatives from set $\hat{B}(Z) \setminus Z$ must be inspected according to their definition.

In this paper it is proved, that due to structure of a GMVS it is enough to inspect the finite number of unfeasible alternatives from set $\hat{B}(Z) \setminus Z$ in order to understand whether this GMVS satisfies intersection property or not. Moreover, insight of what the alternatives should be inspected is provided. These results allow to characterize constructive algorithm of feasibility verification for generalized median voter schemes on compact ranges.

The paper is organized as follows. Section 2 contains notations, definitions and some preliminary results. Section 3 introduces the notation of bricks – Cartesian product ranges, in which any GMVS divides $\hat{B}(Z)$. In Section 4 notation of "direction from an unfeasible alternative to a set of feasible alternatives" is introduced and intersection property is reformulated, using this notation. In Section 5 main theorem is proved, algorithm is provided illustrated by the example of its application in Section 6.

2. PRELIMINARIES: GENERALIZED MEDIAN VOTER SCHEMES AND INTERSECTION PROPERTY

We work with the setting, considered by Barberá, Massó and Serizawa (1998). There are a set of agents $N = \{1,\ldots,n\}$ and a set of coordinates $M = \{1,\ldots,m\}$, $n,m \geq 2$. $Z \subset \mathbb{R}^m$ - a set of feasible alternatives, which is full-dimensional compact. Given $k \in M$, denote projection of Z on k-th coordinate as Z_k, the *minimal box* containing Z:

$$\hat{B}(Z) = \prod_{k \in M} [\min Z_k, \max Z_k].$$

Alternatives from $\hat{B}(Z) \setminus Z$ will be called unfeasible.

Preference u^i of each agent $i \in N$ is continuous complete preorder on alternatives and *multidimensional single-peaked*:

1. it has unique maximal element $\tau(u^i) \in Z$ - the "top" of u^i
2. for any $z, z' \in \mathbb{R}^m$,

$[\, z' \in \hat{B}(\{z, \tau(u^i)\})$ and $z \neq z'\,] \Rightarrow [\, u^i(z') > u^i(z)\,]$.

Let U be domain of all preferences considered. Then *social choice function* (SCF) F is mapping from U^n to Z. Let SCF $F: U^n \to Z$ to be called *manipulable* on U^n if there is exist $u = (u^1,\ldots,u^n) \in U^n$, $i \in N$ and $\tilde{u}^i \in U$ such that $u^i(F(\tilde{u}^i, u^{-i})) \geq u^i(F(u))$, where u^{-i} - utility profile of all agents except i. A SCF $F: U^n \to Z$ is *strategy-proof,* if it is not manipulable on U^n.

According to Barberá, Massó and Serizawa (1998) set of all strategy-proof SCF for the setting considered may be characterized in following way:

A social choice function on the domain of multidimensional single-peak preferences is strategy-proof, iff it is a generalized median voter scheme satisfying intersection property.

In this statement there are two key notations, that we must explain - *generalized median voter scheme* and *intersection property*

Notation of *generalized median voter scheme* (GMVS) was initially introduced by Moulin (1980) for one-dimensional setting. Here we use definition of GMVS in terms of *families of right (left) -coalition systems*, introduced by Barberá, Massó and Neme (1997).

A right (or left) -coalition system on $Z_k \equiv [a_k, b_k]$ is a correspondence W_k that assigns to every $z_k \in Z_k$ a collection $W_k(z_k)$ of coalitions of agents satisfying:

1) Voter sovereignty: $\forall z_k \in (a_k, b_k]$ $([a_k, b_k))$, $W_k(z_k) \neq \varnothing$, $\varnothing \not\subset W_k(z_k)$ and $W_k(a_k) = 2^N \setminus \varnothing$ $(W_k(b_k) = 2^N \setminus \varnothing)$.

2) Coalition monotonicity: if $W \in W_k(z_k)$ and $W \subset W' \Rightarrow W' \in W_k(z_k)$.

3) Outcome monotonicity: if $z'_k < (>)\, z_k$ and $W \in W_k(z_k) \Rightarrow W \in W_k(z'_k)$.

4) Upper semicontinuity: for any $W \in N$, any $z_k \in Z_k$ and any sequence $\{z_k^t\} \subset Z_k$ such that $\lim_{t\to\infty} z_k^t = z_k$, $[\forall t, W \in W_k(z_k^t)] \Rightarrow [W \in W_k(z_k)]$.

A *family* R *of right-coalition systems on* $\hat{B}(Z)$ is a collection $\{R_k\}_{k=1}^m$ where each R_k is a right-coalition system on Z_k. Similarly a *family* L *of left-coalition systems on* $\hat{B}(Z)$ is a collection $\{L_k\}_{k=1}^m$ where each L_k is a left coalition system on Z_k. For each dimension k, let $(\tau_k^1,\ldots,\tau_k^n)$ be the vector of tops projected into this dimension. Then any GMVS is a function $F: U^n \to \hat{B}(Z)$, induced by (Z,R) or (Z,L) in following way: $\forall u \in U^n$, $\forall k \in M$

$F_k(u) = \max\{z_k \in Z_k \mid \{i \in N \mid \tau_k^i \geq z_k\} \in R_k(z_k)\}$ or

$F_k(u) = \min\{z_k \in Z_k \mid \{i \in N \mid \tau_k^i \leq z_k\} \in L_k(z_k)\}$

Given a right-coalition system R_k *corresponding* left-coalition system L_k^* is:

$$L_k^*(z_k) = \{W \in 2^N \mid \forall z'_k > z_k, W' \in R_k(z'_k), W \cap W' = \varnothing\}.$$

Coalition systems R_k and L_k induce same GMVS iff $L_k = L_k^*$.

Due to fact, that any GMVS is defined on $\hat{B}(Z)$ instead of Z it is possible that result, returned by some GMVS for some profile of preferences $u \in U$ will be *unfeasible* - $F(u) \in \hat{B}(Z) \setminus Z$. A GMVS *respects feasibility* if for any $u \in U$ it returns feasible result. It was proved, that any GMVS induced by (Z,R) respects feasibility (or satisfies *intersection property)* iff family of right-coalition systems R has *intersection property* for Z:

A family $R = \{R_k\}_{k=1}^m$ of right-coalition systems on $\hat{B}(Z)$ has the intersection property for Z if for any $y \in \hat{B}(Z) \setminus Z$ and any finite subset $\{z^1,\ldots,z^T\} \subset Z$

$$\bigcap_{t=1}^{T} \left\{ \left[\bigcup_{k \in M^+(y,z^t)} l_k(y_k) \right] \cup \left[\bigcup_{k \in M^-(y,z^t)} r_k(y_k) \right] \right\}. \qquad (1)$$

for every $r_k(y_k) \in R_k(y_k)$ with $k \in \bigcup_{t=1}^T M^-(y, z^t)$ and for every $l_k(y_k) \in L_k(y_k)$ with $k \in \bigcup_{t=1}^T M^+(y, z^t)$, where

$M^{+}(y,x)=\{k\in M: x_k>y_k\}$,
$M^{-}(y,x)=\{k\in M: x_k<y_k\}$.

While in this definition it is necessary to check (1) for any finite subset $\{z^1,\ldots,z^T\}\subset Z$, it was shown by Barberá, Massó and Neme (1997), that for any unfeasible alternative $y\in\hat{B}(Z)\setminus Z$ there is its own unique *crucial set* such that for any family of right-coalition systems it is enough to check (1) in every unfeasible alternative only for its crucial set.

Here we provide not the original definition of crucial set, initially offered by Barberá, Massó and Neme (1997), but its essential for us purposes properties.

A finite subset $S\subset Z$ is crucial for $y\in\hat{B}(Z)\setminus Z$ iff:

1. $\forall x,z\in S$, $x\neq z$ ether $M^{+}(y,x)\not\subset M^{+}(y,z)$ or $M^{-}(y,x)\not\subset M^{-}(y,z)$
2. $\forall z\in Z\setminus T$ $\exists x\in S$ such that $M^{+}(y,x)\subseteq M^{+}(y,z)$ and $M^{-}(y,x)\subseteq M^{-}(y,z)$.

Formally, these results provide full characterization of class of strategy-proof SCFs for the setting considered. But they are not constructive in following sense – definition of intersection property demands to check each unfeasible alternative. And there is infinite number of unfeasible alternatives, because $\hat{B}(Z)\setminus Z$ - continuous set.

3. "BRICKS"

The definition of GMVS in terms of families of right and left coalition systems leads to following – in each dimension $k\in M$ there is exist finite set $\{z_k^1,z_k^2,\ldots,z_k^{W_k}\}\subset Z_k$, $z_k^1=\underline{Z}_k$, $z_k^{W_k}=\overline{Z}_k$ of cardinality $W_k\le 2^n+1$, such that for every $1\le w_k<W_k$:

1. $\forall z_k\in(z_k^{w_k},z_k^{w_k+1})$ $R_k(z_k)=R_k(z_k^{w_k+1})$, $L_k(z_k)=L_k(z_k^{w_k})$;
2. $\forall z_k<z_k^{w_k}$ $R_k(z_k)\subset R_k(z_k^{w_k})$, $L_k(z_k)\supset L_k(z_k^{w_k})$;
3. $\forall z_k>z_k^{w_k}$ $R_k(z_k)\supset R_k(z_k^{w_k+1})$, $L(z_k)\subset L(z_k^{w_k})$.

Elements of these sets may be treated as one-dimensional tops of "phantom" voters according to initial definition of GMVS by Moulin (1980). These sets determined completely by definition of right and left-coalition systems in corresponding dimension. Let us denote $W=\{\ w=(w_1,..,w_k):\forall k\in M\ 1\le w_k<W_k\ \}$. Then Cartesian product of such ranges forms a *brick*: $B_w=\prod_{k=1}^{m}[z_k^{w_k},z_k^{w_k+1}]$.

Any GMVS, defined on $\hat{B}(Z)$, induces on it a set of a bricks $B_W=\{B_w\}_{w\in W}$, such that $\bigcup_{w\in W}B_w=\hat{B}(Z)$.

A brick $B_w\in B_W$ is the *border* brick for Z if $B_w\cap Z\neq\varnothing$ and $B_w\cap\hat{B}(Z)\setminus Z\neq\varnothing$. Let us denote the set of all border bricks for Z via $B_W(cl(Z))$.

We will say, that GMVS *satisfies intersection property for Z in brick* $B_w\in B_W$ if it satisfies intersection property for Z in any unfeasible alternative $y\in B_w\cap\hat{B}(Z)\setminus Z$. Usefulness of bricks for intersection property verification is based on this simple, yet very important results.

<u>Lemma 1.</u> A family $R=\{R_k\}_{k=1}^m$ of right-coalition systems splits $\hat{B}(Z)$ in to set of bricks B_W. Then if for any $B_w\in B_W$, and any $z\in\operatorname{int}B_w$ there are exist: finite set $T\subset\mathbb{N}$, $M_t^+,M_t^-\subseteq M$, $t\in T$, $r_k(y_k)\in R_k(y_k)$, $k\in\bigcup_{t\in T}M_t^+$, $l_k(y_k)\in L_k^*(y_k)$, $k\in\bigcup_{t\in T}M_t^-$ such that

$$\bigcap_{t\in T}\left\{\left[\bigcup_{k\in M_t^-}l_k(y_k)\right]\bigcup\left[\bigcup_{k\in M_t^+}r_k(y_k)\right]\right\}=\varnothing$$

then $\forall z\in B_w$ there are exist $r_k(z_k)\in R_k(z_k)$, $k\in\bigcup_{t\in T}M_t^+$, $l_k(z_k)\in L_k^*(z_k)$, $k\in\bigcup_{t\in T}M_t^-$ such that:

$$\bigcap_{t\in T}\left\{\left[\bigcup_{k\in M_t^-}l_k(z_k)\right]\bigcup\left[\bigcup_{k\in M_t^+}r_k(z_k)\right]\right\}=\varnothing.$$

Proof. It is obvious from the fact that $\forall w\in W\ \forall y,z\in\operatorname{int}B_w$ $R(y)=R(z)$, $L^*(y)=L^*(z)$ and $\forall x\subset clB_w$ $R(y)\subseteq R(x)$, $L^*(y)\subseteq L^*(x)$. Q.E.D.

Lemma 1 results in fact, that it is enough to inspect finite number of alternatives (one from each brick) in order to check whether a GMVS satisfies intersection property for a set of feasible alternatives or not.

The problem that we should solve is which combination of left and right coalitions should be explored for each brick. In original definition of intersection property this combinations was determined by crucial sets for each alternative to be explored. But it is not quite clear, how crucial sets for unfeasible alternatives from one block corresponds to each other. In order to solve this problem we apply notation of a direction, initially introduced in Korgin 2010a.

4. INTERSECTION PROPERTY IN TERMS OF DIRECTIONS

Given m-dimensional Euclidean space $\mathbb{R}^m$, several notations can be introduced.

Direction – m-tuple $d\in 3^M$ where $d_k\in\{-1,0,1\}$, $k\in M$. Using this notation, it is natural to present a direction from one alternative to another in terms *to the left* (-1), *to the right* (1) *and coincides* (0).

Direction $d(y,z)$ *from* $y\in\mathbb{R}^m$ *to* $z\in\mathbb{R}^m$ - is direction where $d_k(y,z)=1$ if $y_k<z_k$, $d_k(y,z)=-1$ if $y_k>z_k$, $d_k(y,z)=0$ if $y_k=z_k$.

Sometimes it will be useful to use notion $d_{-k}(y,z)$ - direction from y to z in all dimensions except $k \in M$. The idea of directions can be expanded in order to determine relative positions of a set of feasible alternatives and single unfeasible alternative in same terms. Let us denote $clZ(y) = \{z \in clZ : \hat{B}(z,y) \cap Z = z\}$.

Definition 1. Direction p is *direction from* $y \in \mathbb{R}^m \setminus Z$ *to* $Z \subset \mathbb{R}^m$, if $\exists z \in clZ(y) : d(y,z) = p$ and if $\exists k \in M : p_k = 0$ then $\neg \exists x \in clZ(y) : d_{-k}(y,x) = p_{-k},\ d_k(y,x) \neq 0$.

It turns out, that according to this definition from an unfeasible alternative there may be more then one direction to Z. That is why let us denote *set of directions from* $y \in \mathbb{R}^m \setminus Z$ *to* $Z \subset \mathbb{R}^m$ - $D(y,Z) = \{p \in 3^M : p$ is *direction from* $y \in \mathbb{R}^m \setminus Z$ *to* $Z \subset \mathbb{R}^m$ $\}$.

Shape of Z defines whether there is $\exists y \in \hat{B}(Z) \setminus Z$ such that $\# D(y,Z) > 1$ or not. Set of feasible alternatives Z is *brick-convex*, if $\forall z, \tilde{z} \in Z \quad \hat{B}(\{z, \tilde{z}\}) \cap Z \neq \{z, \tilde{z}\}$. It is obvious, that any convex compact set is also brick convex. It is quite easy to show, that $\forall y \in \hat{B}(Z) \setminus Z \quad \# D(y,Z) = 1$ iff Z is brick convex.

Given any brick $B_w \in B_W$ such that $B_w \cap \hat{B}(Z) \setminus Z \neq \varnothing$, let us denote the *set of directions from* B_w *to* $Z \subset \mathbb{R}^m$: $D(B_w, Z) = \{D(y,Z)\}_{y \in B_w \cap \hat{B}(Z) \setminus Z}$.

We assume a brick $B_w \in B_W$ such that $B_w \cap \hat{B}(Z) \setminus Z \neq \varnothing$, to be *bad* brick, if $\exists y \in B_w \cap \hat{B}(Z) \setminus Z$ such that $\# D(y,Z) > 1$. Let us denote the set of all bad bricks via $BB_W(Z)$

Using all this notations, definition of the intersection property may be reformulated (in spirit of Nehring and Puppe (2007), using only intersection and no unions) according to following lemma

Lemma 2. A family $R = \{R_k\}_{k=1}^m$ of right coalition systems on $\hat{B}(Z)$ has the intersection property for Z iff $y \in \hat{B}(Z) \setminus Z$, $\forall r_k(y_k) \in R_k(y_k)$, $k \in M^-(y,Z)$, $\forall l_k(y_k) \in L_k^*(y_k)$, $k \in M^+(y,Z)$:

$$\left[\bigcap_{k \in M^+(y,Z)} l_k(y)\right] \cap \left[\bigcap_{k \in M^-(y,Z)} r_k(y)\right] \neq \varnothing, \qquad (2)$$

where $M^+(y,Z) = \{k \in M : \exists d \in D(y,Z),\ d_k = 1\}$ and $M^-(y,Z) = \{k \in M : \exists d \in D(y,Z),\ d_k = -1\}$.

Proof. See Korgin 2010(a).

This formulation of intersection property allows finding out some additional regularities, which help to reduce complexity of verification of the intersection property.

Lemma 3. (*Intersection property monotonisity*) $\forall y \in \hat{B}(Z) \setminus Z : \ \# D(y,Z) = 1$, if a family $R = \{R_k\}_{k=1}^m$ of right coalition systems on $\hat{B}(Z)$ has the intersection property for Z in y, then it will have intersection property in any $x \in \hat{B}(Z) \setminus Z : d(x,y) = d(y,Z)$, $\# D(x,Z) = 1$

Proof. See Korgin 2010(a).

Lemma 3 results in fact, that for all brick-convex ranges it is enough to inspect alternatives close to $cl(Z)$ - just from border bricks.

5. THE CONSTRUCTIVE ALGORITHM OF FEASIBILITY VERIFICATION

All the results above allow formulating main theorem of this paper:

Theorem 1. A family $R = \{R_k\}_{k=1}^m$ of right coalition systems on $\hat{B}(Z)$ has the intersection property for Z iff $\forall B_w \in B_W(cl(Z)) \cup BB_W(Z)$ for any one arbitrary chosen alternative $y \in \operatorname{int} B_w$ holds that: $\forall D \in D(B_w, Z)$ $\forall r_k(y_k) \in R_k(y_k)$, $k \in M^-(D)$, $\forall l_k(y_k) \in L_k^*(y_k)$, $k \in M^+(D)$,

$$\left[\bigcap_{k \in M^+(D)} l_k(y)\right] \cap \left[\bigcap_{k \in M^-(D)} r_k(y)\right] \neq \varnothing, \qquad (3)$$

where $M^+(D) = \{k \in M : \exists d \in D,\ d_k = 1\}$ and $M^-(D) = \{k \in M : \exists d \in D,\ d_k = -1\}$.

Proof. See the Appendix.

This theorem results in fact that while number of unfeasible alternatives is infinite, it is enough to check finite number of alternatives for finite number of directions in order to understand whether a GMVS satisfies intersection property for a set of feasible alternatives or not. And the constructive algorithm of feasibility verification based on this theorem is following:

Step 1. Given GVMS to verify feasibility for set Z, it is necessary to define the set of bricks B_W, produced by this GMVS on $\hat{B}(Z)$.

Step 2. Define the set of border bricks $B_W(cl(Z))$.

Step 3. For each brick from the set of border bricks check, whether given GMVS satisfies intersection property for Z in this brick or not according to theorem 1. If there is exist $B_w \in B_W(cl(Z))$ such that (3) is not true then the algorithm stops – the GMVS under consideration does not satisfy intersection property for Z. In other case algorithm goes to the next step.

Step 4. Define the set of bad bricks $BB_W(Z)$. If it is empty, then algorithm stops – this GMVS satisfies intersection property for Z. In other case algorithm goes to the next step.

Step 5. For each brick from the set of bad bricks check, whether given GMVS satisfies intersection property for Z in this brick or not according to theorem 1. If $\forall B_w \in BB_W(Z)$ (3) is true that this GMVS satisfies intersection property for Z.

For all brick-convex sets of feasible alternatives set of bad bricks will be empty, so the algorithm will stop on step 4.

6. EXAMPLE OF THE ALGORITHM APPLICATION

Let us illustrate algorithm's application with following example which can be characterized as "Problem of resource allocation by voting". There are three projects ($m = 3$), and three agents ($n = 3$), that vote about how the limited amount $C \in \mathbb{R}^1_+$ of resources should be allocated among this three projects while no more than $3C/5$ amount of the resources can be allocated to each project and it is not necessary to spend all the resources available. Top of each agent is the most preferable resource allocation from his point of view. The set of feasible alternatives is (see fig. 1)

$$Z = \{z = \{z_1, z_2, z_3\} \in \mathbb{R}^3_+ \mid z_1 + z_2 + z_3 \in [0, C],. \quad \forall k \in M,\ z_k \in [0, 3C/5]\}$$

Minimal box for this set will be $\hat{B}(Z) = \prod_{k=1}^{3}[0, 3C/5]$.

Graphical representation of Z (hatched area) and $\hat{B}(Z)$ is presented on fig. 1.

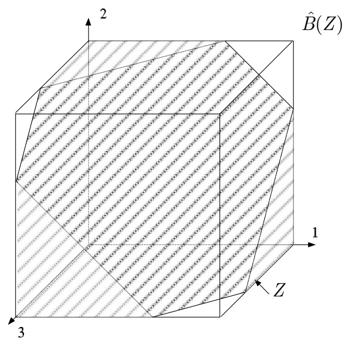

Fig. 1. The set of feasible alternatives Z for example considered and its minimal box $\hat{B}(Z)$.

Let us see, if the GMVS $x = \pi(\tau)$:

$$\forall k \in M\ \pi_k(\tau_k) = \max\{z_k \in [0, \frac{3C}{5}] \mid \mid \#\{i \in N \mid \tau_k^i \geq z_k\} \geq \frac{5nz_k}{3C}\}, \qquad (4)$$

respects feasibility or not for given Z. In case, when GMVS (4) is feasible it turns out that it is the best virtual truthful implementation (see Korgin 2010b) for SCF that calculate mean value of agent's tops $z_k = f(\tau)$:

$$\forall k \in M\ f_k(\tau_k) = \frac{1}{n}\sum_{j=1}^{n}\tau_k^j .$$

Let us apply the algorithm developed for this example

Step 1. The GMVS (4) decomposes $\hat{B}(Z)$ in to the set of bricks $B_W = \{B_w\}_{w \in W}$ where

$$W = \{\ w = (w_1, ..., w_m) : \forall k \in M\ w_k \in \{1, ..., n\}\ \}$$

$$B_w = \prod_{k=1}^{m}[z_k^{w_k}, z_k^{w_k+1}],\ \forall k \in M\ z_k^t = \frac{3C}{5}\frac{t-1}{n}.$$

For this GMVS there is following correspondence between any brick's index (denoted as w) and structure of coalitions that satisfies families of right and left-coalition systems (that generate this GMVS) for any $z \in \operatorname{int} B_w$:

$$z \in \operatorname{int} B_w \Leftrightarrow$$
$$\Leftrightarrow \forall k \in M, \begin{cases} \forall r_k(z_k) \in R_k(z_k)\ \#r_k(z_k) \geq w_k \\ \forall l_k(z_k) \in L_k(z_k)\ \#l_k(z_k) \geq n + 1 - w_k \end{cases} \qquad (5)$$

For case of 3 agents and 3 projects set of bricks B_W, generated by GMVS (4) consists from 27 bricks, and $\forall k \in M\ \ z_k^1 = 0$, $z_k^2 = C/5$, $z_k^3 = 2C/5$, $z_k^4 = 3C/5$.

Step 2. Set of border bricks $B_W(cl(Z))$ in our example consists from bricks $B_w \in B_W$:

$$\sum_{k=1}^{m} z_k^{w_k} \leq C \text{ and } \sum_{k=1}^{m} z_k^{w_k+1} \geq C .$$

It can be easily shown, that $B_w \in B_W(cl(Z)) \Leftrightarrow w_1 + w_2 + w_3 \in \{6, 7\}$. The total number of bricks in $B_W(cl(Z))$ is 13. Graphical representation of $B_W(cl(Z))$ (shaded bricks) is presented on fig. 2.

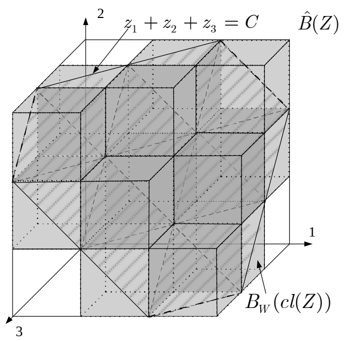

Fig. 2. The set of border bricks.

Step 3. In our example direction from any unfeasible alternative to Z will be "to the left" in each dimension: $\forall y \in \hat{B}(Z) \setminus Z\ D(y, Z) = \{(-1, -1, -1)\}$. Thus, $\forall B_w \in B_W(cl(Z))\quad D(B_w, Z) = \{(-1, -1, -1)\}$. That is mean, that the GMVS considered satisfies intersection property for Z in $B_w \in B_W(cl(Z))$ iff for any one arbitrary chosen alternative $y \in B_w \setminus Z$ holds that:

$\forall r_k(y_k) \in R_k(y_k)$, $k \in M$, $\left[\bigcap_{k\in M} r_k(y)\right] \neq \varnothing$. (6)

It can be easily shown that if a border brick has index w such that $w_1 + w_2 + w_3 = 7$, than it means that $\forall y \in B_w \setminus Z$ $\forall r_k(y_k) \in R_k(y_k)$, $k \in M$ $r_1(y_1) + r_2(y_2) + r_3(y_3) \geq 7$ resulting in fact that (6) holds for this border bricks (for details see Korgin, 2010a)

If a border brick has index w such that $w_1 + w_2 + w_3 = 6$ (6) is not true because $\forall y \in clB_w$ $\forall k \in M$ $\exists r_k(y_k) \in R_k(y_k)$ such that $\# r_k(z_k) = w_k$ and $r_1(y_1) + r_2(y_2) + r_3(y_3) = 6$. It is obvious, that in this case it may be that $r_1(y_1) \cap r_2(y_2) \cap r_3(y_3) = \varnothing$

For example, let us consider unfeasible alternative $(7C/20, 7C/20, 7C/20)$. This alternative belongs to $B_{(2,2,2)}: \frac{1}{5}C < \frac{7}{20}C < \frac{2}{5}C$. This alternative may be chosen by GMVS (4) if at least two agents "vote" for it in each project. Let the tops of agent be: $\tau^1 = (7C/20, 0, 7C/20)$, $\tau^2 = (7C/20, 7C/20, 0)$, $\tau^3 = (0, 7C/20, 7C/20)$. All this tops are feasible and

$\forall k \in M$ $\#\{i \in N \mid \tau_k^i \geq 7C/20\} = 2$. That is why the result according to GMVS (4) is $\pi(\tau) = (7C/20, 7C/20, 7C/20)$.

That is why the GMVS (4) does not satisfy intersection property for the set of feasible alternatives Z considered in this example. That is why the decision rule, based on strategy-proof multiciteria voting rule (4) is not applicable for the problem considered.

Analyzing structure of the set of bricks, generated by the GMVS (4) it can easily be seen, that if the brick structure is such that $\forall k \in M$ $z_k^1 = 0$, $z_k^2 = C/15$, $z_k^3 = C/3$, $z_k^4 = 3C/5$, then the set of border bricks will be $B_W(cl(Z)) = \{B_w \in B_W : w_1 + w_2 + w_3 \geq 7\}$. If the correspondence between brick's indexes and conditions for left and right coalitions inside each brick remains the same as for GMVS (4) (see (5)), then (6) will be satisfied for all the bricks from set of border bricks.

The GMVS that generates such set of bricks is $x = \tilde{\pi}(\tau)$:

$$\forall k \in M \ \tilde{\pi}_k(\tau_k) = \max\{z_k \in [0, \frac{3C}{5}] \mid \\ \mid \#\{i \in N \mid \tau_k^i \geq z_k\} \geq \frac{5nz_k}{4C} + \frac{3}{4}\}.$$

Step 3 will return that all of border bricks satisfies (5) for this GMVS. The algorithm will stop at step 4 in this example because of brick convexity of Z. That is why this GMVS generates strategy-proof decision making mechanism for the problem considered. For example for agents with tops $\tau^1 = (7C/20, 0, 7C/20)$, $\tau^2 = (7C/20, 7C/20, 0)$, $\tau^3 = (0, 7C/20, 7C/20)$ resource allocation will be $\pi(\tau) = (C/3, C/3, C/3)$.

7. CONCLUSIONS

The main result of this paper may be shortly outlined as follows.

First, we show, that in case of infinite number of feasible alternatives it is enough to inspect finite number of unfeasible alternatives in order to understand whether any GMVS respect feasibility or not for concrete set of feasible alternatives.

Second, we provide algorithm for understanding, what alternatives should we inspect, which is "good" in sense that it minimizes number of calculation to be performed in order to understand, whether any GMVS respect feasibility or not for concrete set of feasible alternatives.

Also initial problem, solved in this paper is not so crucial in case of finite number of feasible alternatives (because number of unfeasible alternatives is also finite) algorithm provided will be useful in this case too, because it allows reducing number of alternatives to be inspected.

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APPENDIX

Proof of Theorem 1. Necessity: It follows immediately from Lemma 2.

Sufficiency:

1. From Lemma 1 it is quite easy to show, that in order to understand whether a family $R = \{R_k\}_{k=1}^{m}$ of right coalition systems on $\hat{B}(Z)$ has the intersection property for Z or not it is enough to chose only one alternative from each brick $B_w \in B_W : B_w \cap \hat{B}(Z) \setminus Z \neq \varnothing$ and check (2) for it for each set of directions from $D(B_w, Z)$.

For brick-convex ranges ($BB_W(Z) = \varnothing$) it follows immediately from Lemma 3 that for any unfeasible alternative $y \in \hat{B}(Z) \setminus Z$ there is exist unfeasible alternative from a border brick $x \in \hat{B}(Z) \setminus Z$, such that $d(y,x) = d(x,Z)$. That is why in case of brick-convex ranges it is enough to check only border bricks.

For ranges that are not brick-convex one should consider each unfeasible alternative $y \in \hat{B}(Z) \setminus Z : \# D(y,Z) > 1$ separately. Yet, according to Lemma 1 all such alternatives from one brick have same crucial for feasibility verification properties. That is why it is enough to check (2) for any one alternative from each bad brick and each set of directions from its set of directions. $D(B_w, Z)$. Q.E.D.