ELEMENTS OF THE OPTIMAL SYNTHESIS THEORY FOR FUNCTIONING MECHANISMS OF TWO-LEVEL ACTIVE SYSTEMS. II. SYNTHESIS OF OPTIMAL CORRECT FUNCTIONING MECHANISMS UNDER COMPLETE INFORMATION AT THE HEADQUARTERS

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We derive necessary and sufficient conditions for optimality of correct functioning mechanisms in the case when both the planning procedure for the states of the active elements and the incentive system may be varied. Constructive sufficient conditions of optimality are given for incentive systems ensuring that the plan is met.

1. In [1] we presented a model which described the functioning of a two-level active system and formulated the problem of optimal synthesis of functioning mechanisms ensuring maximum functioning effectiveness of the system under the assumption of complete information available to the central authority (the headquarters). The effectiveness criterion of the functioning mechanism was selected in the framework of modern game-theoretical studies of hierarchic systems: This was the guaranteed estimate of the value of the system's objective function Φ attained on the rational choice set of the lower level active elements. The solution of the optimal synthesis problem in [1] was sought among the so-called correct mechanisms. The main feature of correct mechanisms is that the associated planning procedures and incentive systems ensure that the elements choose precisely the same states as those designated by the headquarters. Necessary and sufficient optimality conditions for correct mechanisms on the set of mechanisms G_π with fixed system objective function Φ and fixed incentive system f were given in [1]. Constructive and easily verified sufficient conditions of optimality of correct mechanisms were also stated. These conditions were presented in the form of constraints on the incentive system.

In this article we continue the study of optimal synthesis of correct functioning mechanisms. The determination of optimality conditions for correct mechanisms considered in [1] is extended to the more general case when both the planning procedure for the states of the active elements and the incentive system for the active elements may be varied in the synthesis problem. We derive necessary and sufficient optimality conditions for correct mechanisms on the set of mechanisms $G_{f,\pi}$ with a fixed system objective function Φ. Constructive sufficient conditions of optimality are defined for incentive systems ensuring that the plan is met.

2. The problem of optimal synthesis of correct functioning mechanisms on a given set of mechanisms $G_{f,\pi}$ has the form [1]

$$K(\hat{\Sigma})=\max_{\Sigma\in G_{f,\pi}} K(\Sigma), \quad \hat{\Sigma}\in G_{f,\pi}\cap \bar{G}_\Sigma. \tag{1}$$

Here and in what follows we use the notation introduced in [1]. We invariably assume that max and min actually exist. It is also assumed that the set of feasible plans X depends on the incentive system f and every plan from this set may be generated using an appropriate planning procedure π. Therefore the expression $\Sigma=\langle\Phi, f, \pi\rangle\in G_{f,\pi}$ in problem (1) may be written in the form $f\in\bar{G}_f$, $x\in X(f)$, where $\bar{G}_f=\{f|\langle\Phi, f, \pi\rangle\in G_{f,\pi}\}$, and the mechanism $\Sigma=\langle\Phi, f, \pi\rangle$ may be equivalently denoted as $\langle\Phi, f, x\rangle$.

Denote by $A(\Sigma)$ the set of efficient plans under the mechanism Σ. This is the set of all plans which, when executed, are no less effective than the plan x under the functioning mechanism $\Sigma=\langle\Phi, f, \pi\rangle$:

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$$A(\Sigma)=\{z\in Y\,|\,\Phi(z,z)\geqslant\min_{y\in R(f,x)}\Phi(x,y)\},$$

where R(f, x) is the set of solutions of the game that the elements play under the hypothesis of locally optimal nonantagonistic behavior [1, 2]. Then the set $A(G_{f,\pi})=\bigcap_{\Sigma\in G_{f,\pi}}A(\Sigma)$ is the set of all efficient plans with respect to the set of mechanisms $G_{f,\pi}$. We will also use the sets $Y(f,x)=\{z|z=x,$ if $x\in R(f,x),$ else $z\in Y\}$ and $S(f)=\{y\in Y|f_i(y_i,y_i)\geqslant f_i(y_i,z_i),z\in Y,\ i\in I\}$, where the latter is the set of perfectly coordinated plans under the incentive system f.

Let us now consider the conditions when the problem (1) has a solution.

THEOREM 1. The problem (1) has a solution if and only if any of the following equivalent conditions holds.

1°. $\exists\hat f\in\overline{G}_f:A(G_{f,\pi})\cap S(\hat f)\cap X(\hat f)\neq\phi;$

2°. $\exists\hat f\in\overline{G}_f,\ \hat x\in A(G_{f,\pi})\cap X(\hat f):\forall y\in Y,\ i\in I:$

$$\hat f_i(\hat x_i,\hat x_i)\geqslant\hat f_i(\hat x_i,y_i); \tag{2}$$

3°. $\exists\hat f\in\overline{G}_f,\ \hat x\in X(\hat f)\cap Y:\forall f\in\overline{G}_f,\ x\in X(f):\exists z\in R(f,x):\forall y\in Y,\ i\in I$ the inequalities (2) and (3) are satisfied:

$$\Phi(\hat x,\hat x)-\Phi(x,z)\geqslant 0; \tag{3}$$

4°. $\exists\hat f\in\overline{G}_f,\ \hat x\in X(\hat f)\cap Y:\forall f\in\overline{G}_f,\ x\in X(f):\exists z\in Y(f,x):\forall y,y'\in Y,\ i\in I$ the inequalities (3) and (3') are satisfied:

$$\hat f_i(\hat x_i,\hat x_i)-\hat f_i(\hat x_i,y_i)\geqslant f_i(x_i,y_i')-f_i(x_i,z_i); \tag{3'}$$

5°. $\exists\hat f\in\overline{G}_f,\ \hat x\in X(\hat f)\cap Y:\forall f\in\overline{G}_f,\ x\in X(f):\exists z\in Y(f,x):\forall y\in Y,\ \alpha,\beta\geqslant 0,\ i\in I$ the inequalities (3) and (3") are satisfied

$$\alpha[\hat f_i(\hat x_i,\hat x_i)-\hat f_i(\hat x_i,y_i)]\geqslant\beta[f_i(x_i,y_i)-f_i(x_i,z_i)]. \tag{3"}$$

The proof is given in the Appendix.

Let us discuss the results of the theorem. Condition 1° is a set-theoretic form of the problem (1). This condition indicates that the problem (1) has a solution if and only if there is a feasible, perfectly coordinated, realizable plan which is no less effective than the other plans. The separation of the properties of perfect coordination, feasibility, and efficiency in condition 1° essentially simplifies the solution of the problem (1). In general, using the principles of coordinated planning, we should start the solution of the corresponding optimal synthesis problem by constructing a condition analogous to 1°.

Let us consider in greater detail the efficiency property.

LEMMA 1. In order to have $A(G_{f,\pi})\neq\phi$,

a) it is necessary and sufficient that $\exists\hat x\in Y:\forall f\in\overline{G}_f,\ x\in X(f):\exists z\in R(f,x)$, such that inequality (3) holds;

b) it is sufficient that the headquarters answer all the losses associated with the deviation of the realizable state y from the plan x: $\Phi(y,y)\geqslant\Phi(x,y)$ for $x\in\bigcup_{f\in\overline{G}_f}X(f),\ y\in\bigcup_{f\in\overline{G}_f}R(f,x)$ (or on the entire set Y).

The lemma is proved in the Appendix.

Condition b) was essentially utilized in the derivation of sufficient optimality conditions for correct mechanisms on the set G_π [1, 2].

The set A(Σ) characterizes the commitment of the headquarters to the application of correct mechanisms under the mechanism Σ. Clearly, if $(G_{f,\pi})=\phi$, the headquarters is not concerned about applying correct mechanisms.

The properties of the set of perfectly coordinated plans S(f) were studied by numerous authors. The bibliography and a summary of the results are given in [2]. Let us present one of the relevant results.

LEMMA 2. If the elements are penalized for failing to meet the plan:

$$\forall x,\ y \in Y,\ i \in I:\ f_i(y_i,\ y_i) \geqslant f_i(x_i,\ y_i), \tag{4}$$

then $S(f) \neq \phi$.

The proof is given in the Appendix.

The conditions 2°-5° of Theorem 1 are alternative forms of the condition 1° which partially (conditions 2° and 3°) or fully (conditions 4° and 5°) recognize the defining inequalities of the sets introduced in condition 1°. This form is largely useful for the derivation of constructive sufficient conditions and for explicit solution of the problem (1).

3. Let us consider some solutions of the problem (1). The form of the solutions actually depends on the properties of the functioning mechanisms on the set $G_{f,\pi}$, i.e., in the final analysis on the properties of the headquarters objective function, the incentive system for the elements, and the model of system constraints [2]. Let us define some of the most useful sets of functioning mechanisms.

The set of mechanisms on which condition b) of Lemma 1 holds will be denoted by G_Σ^1. The sets defined using condition (4) will be denoted by $G_{f,\pi}^2 = \{\langle\Phi,\ f,\ x\rangle \in G_{f,\pi} | R(f,\ x) \cap X(\hat{f}) \neq \phi,$ $\hat{f} \in \overline{G}_f$, and condition (4)} and $G_{f,\pi}^3 = \{\langle\Phi,\ f,\ x\rangle \in G_{f,\pi} | Y \subseteq X(f)$ and condition (4)}. We moreover denote $G_{f,\pi}^4 = \{\langle\Phi,\ f,\ x\rangle \in G_{f,\pi} | \exists \hat{f} \in \overline{G}_f : \forall z \in X(f) \cap Y,\ y \in Y,\ i \in I : \hat{f}_i(z_i,\ z_i) - \hat{f}_i(z_i,\ y_i) \geqslant f_i(x_i,\ z_i) - f_i(x_i,\ y_i)\}$.

We also use the following notation $\overline{G}_f^1 = \{f | \langle\Phi,\ f,\ x\rangle \in G_{f,\pi} \cap G_\Sigma^1\}$, $\overline{G}_f^j = \{f | \langle\Phi, f, x\rangle \in G_{f,\pi}^j\}$ for j = 2, 3, 4 and $\overline{G}_f^{1s} = \overline{G}_f^1 \cap \overline{G}_f^s$, $G_f^{1s4} = \overline{G}_f^{1s} \cap \overline{G}_f^4$, $s = 2, 3$.

The solutions of the problem (1) for these particular sets of mechanisms are presented by the following proposition.

<u>COROLLARY.</u> Conditions 1°-5° hold if one of the following conditions is satisfied.

1^1. $\forall \hat{f} \in \overline{G}_f^1,\ \hat{x} \in X(\hat{f}) \cap Y,\ y \in Y,\ i \in I$:

$$\hat{f}_i(\hat{x}_i,\ \hat{x}_i) - \hat{f}_i(\hat{x}_i,\ y_i) \geqslant 0;$$

2^1. $\exists \hat{f} \in \overline{G}_f^{12} : \forall \hat{x},\ y \in Y \subseteq X(\hat{f}),\ i \in I$:

$$\hat{f}_i(\hat{x}_i,\ \hat{x}_i) - \hat{f}_i(\hat{x}_i,\ y_i) \geqslant \eta_i(\hat{x} - \hat{x}(y_i),\ \Phi(\hat{x},\ \hat{x}) - \Phi(\hat{x},\ \hat{x}(y_i))),$$

where $\eta_i \in \{\varkappa | \varkappa(0,\ 0) = 0$ and $\varkappa(x-y,\ q) \geqslant 0$ for $q \geqslant 0,\ x,\ y \in Y\}$, $\hat{x}(y_i) = (\hat{x}_1,\ \ldots,\ y_i,\ \ldots,\ \hat{x}_n)$;

3^1. $\exists \hat{f} \in \overline{G}_f^{12} : \forall f \in \overline{G}_f^{12},\ \hat{x} \in X(\hat{f}) \cap Y,\ x \in X(f),\ y \in Y,\ i \in I$:

$$\hat{f}_i(\hat{x}_i,\ \hat{x}_i) - \hat{f}_i(\hat{x}_i,\ y_i) \geqslant f_i(x_i,\ z_i) - f_i(x_i,\ y_i);$$

4^1. $\exists \hat{f} \in \overline{G}_f^{12} : \forall \hat{x} \in X(\hat{f}) \cap Y,\ y \in Y,\ i \in I$:

$$\hat{f}_i(\hat{x}_i,\ y_i) = \begin{cases} \max\limits_{f \in \overline{G}_f^{12}} \max\limits_{x \in X(f)} f_i(x_i,\ y_i), & \text{if} \quad \hat{x} = y, \\ \min\limits_{f \in \overline{G}_f^{12}} \min\limits_{x \in X(f)} f_i(x_i,\ y_i), & \text{if} \quad \hat{x} \neq y; \end{cases}$$

5^1. $\exists \hat{f} \in \overline{G}_f^{124} : \forall \hat{x} \in X(\hat{f}) \cap Y,\ y \in Y,\ i \in I$:

$$\hat{f}_i(\hat{x}_i,\ y_i) = \begin{cases} \max\limits_{f \in \overline{G}_f^{124}} f_i(\hat{x}_i,\ y_i), & \text{if} \quad \hat{x} = y, \\ \min\limits_{f \in \overline{G}_f^{124}} f_i(\hat{x}_i,\ y_i), & \text{if} \quad \hat{x} \neq y. \end{cases}$$

The proof is given in the Appendix.

<u>Remark 1.</u> The results presented in the above Corollary remain valid if in conditions 1^1-5^1 the set $\overline{G}_f^2$ is replaced with $\overline{G}_f^3$. In this case, the expression $\forall \hat{x} \in X(\hat{f}) \cap Y$ should be replaced with $\forall \hat{x} \in Y$. Some solutions of the problem (1) for the case with the set $\overline{G}_f^3$ were previously considered in [2].

<u>Remark 2.</u> The optimality conditions of correct mechanisms as derived in [1] for the case $G_\Sigma^3 = G_\pi \subseteq G_\Sigma^1$ in general are only necessary for the problem (1) considered in this article. But if the problem (1) from [1] is solved for $\forall G_\pi \subseteq G_{f,\pi}$, then the resulting solutions will include the solution of the problem (1) in this article. An example of such a solution is provided by condition 1^1.

Using condition 2^1, we can construct an optimal incentive system $\hat{f}$ in which the system objective function Φ is used as the "prototype." Note that if either of the conditions 1^1 or 2^1 holds, then appropriate choice of the planning procedure ensures absolutely optimal functioning of the system [2].

It follows from condition 4^1 than one of the optimal strategies of the headquarters is the strategy which prescribes maximum incentives when the plan is met and minimum incentives when the plan is not met.

Let us represent the objective functions of the elements f_i, $i\in I$ in the form $f_i(x_i, y_i)=$ $=h_i(y_i)-\chi_i(x_i, y_i)$, $i\in I$ for the incentive system $f = (h, \chi)$ [2], where $h_i(y_i) = f_i(y_i, y_i)$, $\chi_i(x_i, y_i) = h_i(y_i) - f_i(x_i, y_i)$ is the penalty function for failure to meet the plan. Let $\bar{G}_\chi^{12}(h)=\{\chi|(h, \chi)\in\bar{G}_f^{12}\}$ and $\bar{G}_\chi^{124}(h)=\{\chi|(h,\chi)\in\bar{G}_f^{124}\}$. Consider the conditions 3^1-5^1 for the case when h is fixed. After appropriate transformations, we obtain

3^2. $\exists\hat{\chi}\in\bar{G}_\chi^{12}(h):\forall\chi\in\bar{G}_\chi^{12}(h),\ \acute{x}\in X(h, \hat{\chi})\cap Y,\ x\in X(h, \chi),$

$$y\in Y,\ i\in I:\ \hat{\chi}_i(\acute{x}_i, y_i)\geqslant\chi_i(x_i, y_i)-\chi_i(x_i, \hat{x}_i);$$

4^2. $\exists\hat{\chi}\in\bar{G}_\chi^{12}(h):\forall z\in X(h, \hat{\chi})\cap Y,\ y\in Y,\ i\in I:$

$$\hat{\chi}_i(\hat{x}_i, y_i)=\begin{cases}0, & \text{if}\quad \hat{x}_i=y_i,\\ \max\limits_{\chi\in\bar{G}_\chi^{12}(h)}\ \max\limits_{x\in X(h,\chi)}\chi_i(\hat{x}_i, y_i), & \text{if}\quad \hat{x}_i\neq y_i;\end{cases}$$

5^2. $\exists\hat{\chi}\in\bar{G}_\chi^{124}(h):\forall\acute{x}\in X(h,\hat{\chi})\cap Y,\quad y\in Y,\quad i\in I:$

$$\hat{\chi}_i(\acute{x}_i, y_i)=\max_{\chi\in\bar{G}_\chi^{124}(h)}\chi_i(\acute{x}_i, y_i).$$

As in the Corollary, we may replace the set $\bar{G}_\chi^2(h)$ with the set $\bar{G}_\chi^3(h)$ in the conditions 3^2-5^2.

The condition 3^2 may be considered as a generalization of the sufficient condition of strong coordination [2] to the case when only Φ and h are fixed. If the function χ is also fixed, this condition reduces to the condition of strong coordination 4°b in [1].

Condition 5^2 implies that the optimal incentive system on the set $\bar{G}_\chi^{124}(h)$ is the one with maximum degree of centralization [2].

Conditions 3^2-5^2 may also be used in cases when h is variable. To this end it suffices to ensure that the penalty functions $\hat{\chi}_i$, $i\in I$ have an additional jump at the point z = y with the magnitude

$$\Delta_i=\max_{y\in Y}[\hat{h}_i(y_i)-h_i(y_i)]-\min_{y\in Y}[\hat{h}_i(y_i)-h_i(y_i)].$$

An appropriate penalty function in this case is

$$\hat{\chi}_i'(\hat{x}_i, y_i)=\begin{cases}0, & \text{if}\quad \hat{x}_i=y_i,\\ \hat{\chi}_i(\acute{x}_i, y_i)+\Delta_i, & \text{if}\quad \hat{x}_i\neq y_i.\end{cases}$$

4. Let us consider an example of optimal synthesis of the penalty function for the active element in a system which consists of the headquarters and a single element. Let the state of the element be specified by the scalar y; the set of states by the set $Y=\{y|a\leqslant y\leqslant b\}$; the plan by the scalar x; the set of feasible plans by the set $X_s=\{v|c_s\leqslant x\leqslant d_s\}$, where s is an integer parameter of the penalty function χ_s. The set of penalty functions has the form

$$\bar{G}_\chi(h)=\{(q(z, y)-y)^{2s}-(q(z, y)-z)^{2s}|\ s=\overline{1, m},\ y\in Y,\ z\in X_s\cap Y\},$$

where q is some function with values in X_s. The form of the function h does not affect the solution of the synthesis problem.

We will solve our problem using the condition 3^2. We define the set

$$\bar{G}_\chi^{13}(h)=\{((q(z, y)-y)^{2s}-(q(z, y)-z)^{2s}\geqslant 0)\,|\,s=\overline{1, m},\ y, z\in Y\subseteq X_s\}.$$

We stipulate that $\bar{G}_\chi^{13}(h)\neq\phi$. This is possible if $c_s\leqslant a\leqslant b\leqslant d_s$ and $y\leqslant z\leqslant q(z, y)\leqslant d_s$, or if $c_s\leqslant q(z, y)\leqslant z\leqslant y$ for $s = \overline{1, m}$.

The function q is defined by the equality

$$(q(z,y)-y)^{2s}-(q(z,y)-z)^{2s}=\max_{x\in X_s}[(x-y)^{2s}-(x-z)^{2s}].$$

For $x \leqslant z \leqslant y$ and $y \leqslant z \leqslant x$ the x-derivative of $(x-y)^{2s}-(x-z)^{2s}$ for all z, $y \in Y$ has a constant sign. Therefore the function $(x-y)^{2s}-(x-z)^{2s}$ in this case attains its maximum on the boundary of the set X_s, i.e., $q(z, y) = \{c_s \text{ if } y \geqslant z, \text{ otherwise } d_s\}$. The solution of our problem is thus sought among penalty function having the form

$$\chi_s(z,y)=\sum_{j=0}^{2s} C_{2s}^j q^{2s-j}(z,y)(-1)^j(y^j-z^j),$$

where $C_{2s}^j=(2s)!/j!(2s-j)!$.

The optimal value of s depends on the choice of X_s and Y. Let us consider some alternatives. Let

$$c_s+\frac{1}{c_s-p}\leqslant a\leqslant b\leqslant d_s-\frac{1}{d_s+p},\quad p=\sqrt{\frac{s-1}{s}}. \tag{5}$$

Then we can show that $\chi_s(z,y)-\chi_{s-1}(z,y)\geqslant 0$ for $\forall z, y\in Y$, if additionally $c_s\leqslant c_{s-1}$ and $d_s\geqslant d_{s-1}$. Therefore, from the condition 3° it follows that the optimal value of s is s = m. In this case condition (5) defines the appropriate family of sets of feasible plans X_s, $s = \overline{1, m}$.

Let $c_s\leqslant a\leqslant d_s-\frac{1}{d_s+p}\leqslant c_s+\frac{1}{c_s-p}\leqslant b\leqslant d_s$. Then, conversely, $\chi_{s-1}(z,y)\geqslant\chi_s(z,y)$ for $\forall z, y\in Y$ subject to the additional constraints $c_s\geqslant c_{s-1}$ and $d_s\leqslant d_{s-1}$. As a result, the optimal s is equal to 1.

These solutions in general do not exhaust all the possible solutions. We have not considered the case when the optimal value of s is attained on the boundary of the set $\overline{1, m}$ and the case when s depends on z, y. The latter is the most general case, but the corresponding penalty function is much too complex to allow a constructive approach and is therefore not considered here.

APPENDIX

If condition 1° implies 2°, we write 1° = > 2°.

Proof of Theorem 1.

(1) = >1°. Assume the contrary. Then $\forall\hat f\in\bar G_f: A(G_{f,\pi})\cap S(\hat f)\cap X(\hat f)=\phi$ or $\exists\tilde\Sigma\in G_{f,\pi}:\forall f\in\bar G_f$: $A(\tilde\Sigma)\cap S(\hat f)\cap X(\hat f)=\phi$. From the definition of A(Σ) it follows that $K(\Sigma')\geqslant K(\Sigma)$, if $A(\Sigma')\subseteq A(\Sigma)$, and $K(\Sigma') > K(\Sigma)$, if $A(\Sigma)\setminus A(\Sigma')\neq\phi$ for $\Sigma, \Sigma'\in G_{f,\pi}$. By assumption, any plan from $X(\hat f)$ may be adopted. Therefore, take the optimal perfectly coordinated planning procedure [1, 2] which we denote by $\hat\pi$. Then for $\hat\Sigma=\langle\Phi, \hat f, \hat\pi\rangle\in G_{f,\pi}$ we have $A(\hat\Sigma)\cap S(\hat f)\cap X(\hat f)\neq\phi$. But since $A(\tilde\Sigma)\cap S(\hat f)\cap X(\hat f)=\phi$, we obtain $A(\hat\Sigma)\setminus A(\tilde\Sigma)\neq\phi$ and $K(\hat\Sigma) < K(\tilde\Sigma)$, which contradicts condition (1). Thus (1) implies 1°.

1° = >(1). Take the plan $\hat x\in A(G_{f,\pi})\cap X(\hat f)\cap S(\hat f)$. This is clearly an optimal perfectly coordinated plan. Therefore $A(\langle\Phi, \hat f, \hat x\rangle)=A(G_{f,\pi})\subseteq A(\Sigma)$ for $\forall\Sigma\in G_{f,\pi}$. Hence it follows that $K(\hat\Sigma)\geqslant K(\Sigma)$ for $\forall\Sigma\in G_{f,\pi}$. Q.E.D.

1° =>2°. Let $\hat x\in A(G_{f,\pi})\cap S(\hat f)\cap X(\hat f)$; then for $\forall y\in Y$, $i\in I$ inequality (2) holds and $\hat x\in A(G_{f,\pi})\cap X(\hat f)$. QED.

2° = > 1°. Since $\hat x\in A(G_{f,\pi})\cap X(\hat f)$ and for $\forall y\in Y$, $i\in I$ inequality (2) holds, $\hat x$ belongs to both sets $A(G_{f,\pi})\cap X(f)$ and $S(\hat f)$. Therefore $A(G_{f,\pi})\cap X(\hat f)\cap S(\hat f)\neq\phi$.

2° = >3°. Since $\hat x\in A(G_{f,\pi})\cap X(\hat f)$, we obtain

$$\Phi(\hat x,\hat x)\geqslant\min_{y\in R(f,x)}\Phi(x,y)$$

for $\forall\langle\Phi, f, x\rangle\in G_{f,\pi}$. But for $\forall\langle\Phi, f, x\rangle\in G_{f,\pi}:\exists z\in R(f,x):\Phi(x,z)=\min_{y\in R(f,x)}\Phi(x,y)$.

Substituting this equality in the previous inequality, we obtain (3) and thus the condition 3°.

$3° => 2°$. Since $z \in R(f, x)$, inequality (3) gives

$$\Phi(\hat{x}, \hat{x}) \geqslant \Phi(x, z) \geqslant \min_{y \in R(f,x)} \Phi(x, y)$$

for $\forall \langle \Phi, f, x \rangle \in G_{f,\pi}$. From the definition of the set $A(\Sigma)$ and this inequality it follows that

Therefore condition 2° holds.

$3° => 4°$. Since $z \in R(f, x)$, we obtain $\forall y' \in Y,\ i \in I$: $f_i(x_i, z_i) - f_i(x_i, y_i') \geqslant 0$. Adding this inequality to inequality (2), we obtain the missing inequality of condition 4° and thus prove the condition itself.

$4° => 3°$. Assume the contrary. Then $\forall \hat{f} \in \bar{G}_f,\ \hat{x} \in X(\hat{f}) \cap Y : \exists f \in \bar{G}_f,\ x \in X(f) : \forall z \in R(f, x) : \exists y \in Y,\ i \in I,$ such that inequality (2) is violated. Since condition 4° holds for $\forall y' \in Y$, we take $y' \in R(f, x)$; then for $\forall i \in I,\ x \in X(f),\ z \in Y$ we have the inequality $f_i(x_i, y_i') \geqslant f_i(x_i, z_i)$. Add this inequality with the inequality $\hat{f}_i(\hat{x}_i, \hat{x}_i) < \hat{f}_i(\hat{x}_i, y_i)$, which follows from (2) in the light of our assumption. We obtain $\hat{x} \in A(G_{f,\pi})$.

$$\hat{f}_i(\hat{x}_i, \hat{x}_i) - \hat{f}_i(\hat{x}_i, y_i) < f_i(x_i, y_i') - f_i(x_i, z_i).$$

This inequality contradicts the second inequality of condition 4°, i.e., our assumption regarding inequality (2) is not true. But then in condition 4° $z \in R(f, x)$ and inequality (3) also should hold. QED.

The implications $3° => 5°$ and $5° => 3°$ are proved along the same lines as $3° => 4°$ and $4° => 3°$. QED.

Proof of Lemma 1. Consider the condition a).

Necessity. Since $A(G_{f,\pi}) \neq \phi$, take $\hat{x} \in A(G_{f,\pi})$. Then

$$\Phi(\hat{x}, \hat{x}) \geqslant \min_{y \in R(f,x)} \Phi(x, y)$$

for $\forall \langle \Phi, f, x \rangle \in G_{f,\pi}$ or $\forall f \in \bar{G}_f,\ x \in X(f)$. But $\exists z \in R(f, x)$, such that $\Phi(x, z) = \min \Phi(x, y)$ over $y \in R(f, x)$. Substituting this equality in the previous inequality, we get the inequality (3) and thus condition a).

Sufficiency. Assume the contrary. Then $A(G_{f,\pi}) = \phi$. In this case, from the definition of the set $A(\Sigma)$ we obtain $\forall \hat{x} \in Y : \exists f \in \bar{G}_f,\ x \in X(f) : \forall z \in R(f, x)$, such that inequality (3) is violated. This contradicts condition a). The contradiction proves sufficiency.

Consider the sufficiency of condition b). Assume the contrary. Then, repeating the first part of the preceding proof, we conclude that inequality (3) is a fortiori violated if we replace $\exists f \in \bar{G}_f,\ x \in X(f) : \forall z \in R(f, x)$ with $\forall x \in \bigcup_{f \in \bar{G}_f} X(f),\ z \in \bigcup_{f \in \bar{G}_f} R(f, x)$ (or $z \in Y$) and take $\hat{x} = z$. But then we end up with a condition which contradicts condition b). The contradiction completes the proof of sufficiency of condition b) and Lemma 1. QED.

Proof of Lemma 2. Assume the contrary. Then $S(f) = \phi$ and from the definition of the set $S(f)$, $\forall z \in Y : \exists i \in I$:

$$\max_{y \in Y} f_i(z_i, y_i) > f_i(z_i, y_i).$$

Take $y' \in R(f, z)$, then

$$\max_{y \in Y} f_i(z_i, y_i) = f_i(z_i, y_i') > f_i(z_i, z_i),$$

which contradicts the assumption of Lemma 2. QED.

Proof of Corollary.

$1^1 => 2°$. Assume the contrary. Then $\forall \hat{f} \in \bar{G}_f,\ \hat{x} \in A(G_{f,\pi}) \cap X(\hat{f}) : \exists y \in Y,\ i \in I$, such that inequality (2) is violated. From condition b) of Lemma 1 it follows that for $G^1_{f,\pi} = G_{\hat{x}}{}^1 \cap G_{f,\pi}$ the set $A(G^1_{f,\pi}) \neq \phi$ and since by condition 1^1 $X(\hat{f}) \cap Y \neq \phi$, also $A(G^1_{f,\pi}) \cap X(\hat{f}) \neq \phi$. Therefore the inequality is a fortiori violated if we replace $\forall \hat{f} \in \bar{G}_f,\ x \in A(G_{f,\pi}) \cap X(\hat{f})$ with $\exists \hat{f} \in \bar{G}_f{}^1,\ \hat{x} \in X(\hat{f}) \cap Y$. This is a contradiction of condition 1^1. QED.

$2^1 => 3°$. Assume the contrary. Then $\forall \hat{f} \in \bar{G}_f,\ \hat{x} \in X(\hat{f}) \cap Y : \exists f \in \bar{G}_f,\ x \in X(f) : \forall z \in R(f, x) : \exists y \in Y,\ i \in I,$ such that one or simultaneously both inequalities in 3° are violated.

Since in condition 2^1, $\hat{f}\in\bar{G}_f^{12}$, the system objective function Φ satisfies the condition b) of Lemma 1. Therefore it suffices to take $z = \hat{x}$ in order to verify that inequality (3) holds.

Consider inequality (2). Take $\hat{x}\in Y$, such that $\Phi(\hat{x}, \hat{x}) - \Phi(x, x) \geqslant 0$ for $\forall x\in Y$. Using this and condition b) of Lemma 1, we obtain $\Phi(\hat{x}, \hat{x}) \geqslant \Phi(\hat{x}(y_i), \hat{x}(y_i)) \geqslant \Phi(\hat{x}, \hat{x}(y_i))$ for $\forall y\in Y$. Hence $\eta_i(\hat{x}-\hat{x}(y_i), \Phi(\hat{x}, \hat{x}) - \Phi(\hat{x}, \hat{x}(y_i))) \geqslant 0$. But by condition 2^1, $\hat{f}_i(\hat{x}_i, \hat{x}_i) - \hat{f}_i(\hat{x}_i, y_i) \geqslant 0$, which contradicts our assumption and proves sufficiency of 2^1.

$3^1 => 5°$. Assume the contrary. Then $\forall \hat{f}\in\bar{G}_f$, $\hat{x}\in X(\hat{f})\cap Y : \exists f\in\bar{G}_f$, $x\in X(f) : \forall z\in Y(f, x) : \exists y\in Y$, $\alpha, \beta\geqslant 0$, $i\in I$:

$$\alpha\,[\hat{f}_i(\hat{x}_i, \hat{x}_i) - \hat{f}_i(\hat{x}_i, y_i)] < \beta\,[f_i(x_i, y_i) - f_i(x_i, z_i)]. \tag{6}$$

Since sufficiency of 3^1 is considered on the set $\bar{G}_f^{12}$, the objective function Φ satisfies condition b) of Lemma 1. Therefore the inequality (3) holds.

The strict inequality (6) should also hold for $z\in R(f, x)$. But then $f_i(x_i, z_i) \geqslant f_i(x_i, y_i)$ and as a result (6) implies that $\hat{f}_i(\hat{x}_i, \hat{x}_i) < \hat{f}_i(\hat{x}_i, y_i)$. In this case the inequality (6) is a fortiori true if we take $\alpha = -\beta \neq 0$. But the resulting inequality contradicts the inequality in 3^1. This contradiction proves the sufficiency of condition 3^1.

$3^1 => 4^1$. From the inequality of condition 3^1 we obtain

$$\hat{f}_i(\hat{x}_i, \hat{x}_i) - \hat{f}_i(\hat{x}_i, y_i) \geqslant \max_{f\in\bar{G}_f^{13}} \max_{z\in X(f)} [f_i(z_i, x_i) - f_i(z_i, y_i)] \tag{7}$$

for $\forall \hat{x}\in X(\hat{f})\cap Y$, $y\in Y$, $i\in I$. Clearly,

$$\max_{f\in\bar{G}_f^{13}} \max_{z\in X(f)} [f_i(z_i, x_i) - f_i(z_i, y_i)] \leqslant$$
$$\leqslant \max_{f\in\bar{G}_f^{13}} \max_{z\in X(f)} f_i(z_i, x_i) - \min_{f\in\bar{G}_f^{13}} \min_{z\in X(f)} f_i(z_i, y_i)$$

for $\forall i\in I$. Comparing this inequality with (7), we obtain the desired result.

$3^1 => 5^1$. Substituting the defining inequality of the set $\bar{G}_f^4$ in (6), we obtain the desired result.

QED.

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