OPTIMAL MECHANISMS FOR PLANNING AND STIMULATION IN ACTIVE SYSTEMS

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Active systems with incomplete information at the center about the set of possible states of active elements are considered. Optimal stimuli and planning procedures depending on the reporting of active elements to the information center are found.

1. INTRODUCTION

The problems of constructing optimal planning and stimulation mechanisms ("functional mechanisms") are central problems in the theory of active systems [1, 2], and have been studied in many papers. At present the problems of synthesizing optimal functional mechanisms with complete information at the center about models of active elements [2, 3], synthesis of optimal stimuli in risk conditions [4], and synthesis of optimal procedures for a fixed system of stimuli with incomplete information at the center [5, 6], have been solved. The problem of synthesizing optimal planning and stimulation mechanisms with incomplete information at the center in an active system with exchange of information is practically untouched. The exception is papers on game theory with nonopposed interests, where this problem is investigated in order to guarantee a strategy with sufficiently strong assumptions on the "caution" of the elements [7, 8].

Here we find an optimal planning and stimulation mechanism for a system with one active element (corresponding, for example, to a monopoly industry in an economic system) and exchange of information between active elements (AE) and the center under some very natural assumptions. A number of concepts used here are described in more detail in [1, 2].

2. A MODEL OF AN ACTIVE SYSTEM AND STATEMENT OF THE PROBLEM

We consider an active system consisting of a center and active elements (AE). Let $\varphi(x, r)$ and $\Psi(x, r)$ be preference functions for the AE and the center respectively, defined on a set of allowed goals $X = [x^\ell, x^h]$ and a set $A = [r^\ell, r^h]$ of possible values of a parameter r. We assume that the value of r is known at an active element, that only the set A is known at the center, and that the AE reports an estimate $s \in A$ of r to the center. We assume also that the preference function of the AE may be written in the form $\varphi(x, r) = \sigma(x) - \zeta(x, r)$ where $\sigma(x)$ is a stimulus, and $\zeta(x, r)$ is a cost function for the AE.

We define $\pi(\cdot)$ to be the planning procedure, viewed as a representation of the set A of allowed reports in set X.

The problem of constructing optimal functional mechanisms $\mu^* = (\sigma^*(\cdot), \pi^*(\cdot))$ is formulated in the following way: on the set M of allowed mechanisms find a mechanism μ^*, such that

$$K(\mu^*) = \max_{\mu \in M} K(\mu), \tag{1}$$

where

$$K(\mu) = \min_{r \in A} [\min_{s \in R(r)} \Psi(\pi(s), r)/\Psi_w(r)]$$

is a measure of the performance, and M is a set of allowed mechanisms, $\mu = (\sigma(\cdot), \pi(\cdot))$. Here $R(r) = \operatorname{Arg}\max_{s \in A} \varphi(\pi(s), r)$; $\Psi_w(r)$ is a weighing function, for example $\Psi_w(r) = \max_{x \in X} \Psi(x, r)$.

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From here onwards it is assumed that the max and min operations are defined.

The Principle of Open Control (OC) and Optimal Planning Procedures [5, 6]. For a system with one AE an open control (planning) procedure is defined as a mapping $\pi^{OC}(\cdot)$, satisfying the "perfect agreement" conditions

$$\forall s \in A: \ \varphi(\pi^{OC}(s), s) = \max_{x \in X_c} \varphi(x, s), \tag{2}$$

where X_c is a given subset of X.

We note some properties of open control mechanisms [5].

1. The class of open control procedures contains optimal planning procedures.

2. The procedure $\pi^{OC}(\cdot)$, stimulates reporting of accurate information, i.e., $\forall s, r \in A$: $\varphi(\pi^{OC}(s), r) \le \varphi(\pi^{OC}(r), r)$.

3. The index of performance of the functional mechanism with an open control procedure has the form $K(\mu) = \min_{r \in A} (\Psi(\pi^{OC}(r), r)/\Psi_w(r))$ with satisfaction of the condition of "noncounteracting AE" which is formulated as follows: $R(r) = \{r\}$, if $r \in R(r)$.

We make a number of assumptions. Let the cost function $\zeta(x, r)$ be twice differentiable in x and r and $\dot{\zeta}_x(x, r) > 0$, $\ddot{\zeta}_{xx}(x, r) > 0$, $\ddot{\zeta}_{xr}(x, r) < 0$ for all $x \in X$, $r \in A$. Furthermore the set of allowed mechanisms has the form $M = \{\mu \mid 0 \le \sigma(x) \le g, x = \pi(s), x \in X, s \in A\}$, where g is a given constant, determining the limits on the size of the stimulus $\Psi(x, r)$, $\Psi_w(r)$, is a continuous function and $\Psi(x, r) \ge 0$, $\Psi_w(r) > 0$ for $x \in X$, $r \in A$.

3. CONSTRUCTION OF OPTIMAL STIMULUS MECHANISMS

Suppose some nondecreasing, possibly multivalued function $\pi(s)$ is defined on A, i.e., such that

$$\forall s_1, s_2 \in A, \quad s_1 < s_2: \quad \pi(s_1) \leqslant \pi(s_2). \tag{3}$$

We note that it follows from the properties of monotonic functions that the number of points where $\pi(s)$ is multivated is at most countable.

We assume that the graph of $\pi(s)$ is a connected set $\Gamma = \{(s, \pi(s)) \mid s \in A\}$. In this case there is a nondecreasing function $\tilde{r}(x)$, possibly multivalued, at most at a countable number of points, inverse to $\pi(s)$, representing the section $[\pi(r^{\ell}), \pi(r^{h})]$ on A. The graph of the function $\tilde{r}(x)$, obviously, also represents the same connected set. We will say a function $\pi(s)$ satisfying these conditions is a basis function.

The basis function $\tilde{\pi}(s)$ gives a class of basic planning procedures $\pi(s)$, coinciding with $\tilde{\pi}(s)$ at all points where $\tilde{\pi}(s)$ is single valued, and taking one of the values of $\tilde{\pi}(s)$ at the points s where $\tilde{\pi}(s)$ is multivalued.

For the given basic planning procedures $\pi(s)$ we consider the following problem: find a stimulus function $\tilde{\sigma}_\pi(x)$ such that for this stimulus function the procedure $\pi(s)$ is an open control procedure with $X_c = [\pi(r^{\ell}), \pi(r^{h})]$.

It is easy to see that, with the substitution $\varphi(x, r) = \tilde{\sigma}_\pi(x) - \zeta(x, r)$ in (2), the solution of this problem has the form

$$\tilde{\sigma}_\pi(x) = C + \begin{cases} 0, & \text{if} \quad x \leqslant \pi(r^{\ell}), \\ \int_{\pi(r^{\ell})}^{x} \dot{\zeta}_t(t, \tilde{r}(t))\,dt, & \text{if} \quad \pi(r^{\ell}) < x < \pi(r^{h}), \\ \int_{\pi(r^{\ell})}^{\pi(r^{h})} \dot{\zeta}_t(t, \tilde{r}(t))\,dt, & \text{if} \quad x \geqslant \pi(r^{h}), \end{cases} \tag{4}$$

where $\tilde{r}(x)$ is an inverse of $\pi(s)$ and C is an arbitrary constant.

We say a stimulus function $\tilde{\sigma}_\pi(x)$ corresponds to the basic planning procedure $\pi(s)$, if (4) holds.

We fix some values of the performance index $K(\mu) = \gamma$ and consider the set of planning procedures $x = \pi_\gamma(x)$ such that all pairs (x, r) satisfy

$$\forall r \in A, \quad x = \pi_\gamma(r): \quad \Psi(x, r) \geqslant \gamma \Psi_W(r). \tag{5}$$

This set is nonempty, at least for small enough γ.

We denote by L_γ the set of basic planning procedures $\pi(r)$, satisfying (5) for a given γ.

We note that $L_{\gamma_1} \subseteq L_{\gamma_2}$ if $\gamma_1 > \gamma_2$.

We state the problem of finding, in the class L_γ, functions $\pi(\cdot)$, which correspond to "minimal" stimulus functions. This problem is formulated as follows.

Problem A. Find a procedure $\pi^*(\cdot) \in L_\gamma$, such that

$$\sigma(\pi^*(r^h)) = \min_{\pi(\cdot)\in L_\gamma} \sigma(\pi(r^h)). \tag{6}$$

Solution of problem A allows determination of basis planning procedures $x = \pi^*(r)$ and corresponding stimulus functions σ, which guarantee the value of the performance index for the functional mechanism $K(\mu)$ is not less than γ. In this case, in agreement with (6) the maximal value of the stimulus $\sigma(\pi^*(r^h))$ is minimal.

We note that from $L_{\gamma_1} \subseteq L_{\gamma_2}$ for $\gamma_1 > \gamma_2$ it follows that

$$\tilde{\sigma}(\pi^*(r^h_{\gamma_2})) \geqslant \tilde{\sigma}(\pi^*(r^h_{\gamma_1})).$$

We also consider the following problem.

Problem B. Find γ^* and $\pi^*_{\gamma^*}(r)$, such that

$$\tilde{\sigma}(\pi^*_{\gamma^*}(r^\ell)) = 0, \quad \tilde{\sigma}(\pi^*_{\gamma^*}(r^h)) = \tilde{g}, \tag{7}$$

where

$$\tilde{g} = \begin{cases} g, & \text{if} \quad g \leqslant \tilde{\sigma}(\pi^*_{\gamma^M}(r^h)), \\ \tilde{\sigma}(\pi^*_{\gamma^M}(r^h)), & \text{if} \quad g > \tilde{\sigma}(\pi^*_{\gamma^M}(r^h)), \end{cases} \quad \gamma^M = \max\{\gamma \mid L_\gamma \neq \phi\}.$$

The following theorem establishes a link between the initial problem (1) and problem B.

THEOREM 1. The solution of problem B determines the solution of problem (1), moreover

$$\begin{aligned} &K(\mu^*) = \gamma^*, \\ &\pi^*(r) = \pi^*_{\gamma^*}(r), \\ &\tilde{\sigma}^*(x) = \int_{\pi_{\gamma^*}(r)}^{x} \dot{\zeta}_t(t, \tilde{r}_{\gamma^*}(t))\,dt, \end{aligned} \tag{8}$$

where $\tilde{r}_{\gamma^*}(t)$ is a function inverse to $\pi_{\gamma^*}(t)$.

Proof of the theorem is given in the appendix.

The theorem implies that problem (1) reduces to solution of problem B in the class of basis functions satisfying (5).

We assume that $\Psi(x, r)$ is quasiconcave with respect to x and investigate the solution of problem B.

Lemma 1. If the function $\Psi(x, r)$ is quasiconcave in x and γ is such that (5) is soluble $\forall r \in A$, then the set of points (x, r), determined by (5) may be written in the form $\{(x, r) \mid q_1(\gamma, r) \le x \le q_2(\gamma, r), r \in A\}$ where $q_1(\gamma, r)$, and $q_2(\gamma, r)$ is a function whose graph is a connected set.

Proof of the lemma is given in the appendix.

We now consider the function

$$\underline{q}_2'(\gamma, r) = \min_{r \leqslant p \leqslant r^h} q_2(\gamma, p) \quad \text{and} \quad \bar{q}_1'(\gamma, r) = \max_{r^\ell \leqslant p \leqslant r} q_1(\gamma, p).$$

The min and max operations are well defined since $q_1(\gamma, r)$ and $q_2(\gamma, r)$ are functions with connected graphs. We note that $\underline{q}_2'(\gamma, r)$ and $\bar{q}'_1(\gamma, r)$ are nondecreasing functions.

If the function $\underline{q}_2'(\gamma, r)$ and $\bar{q}_1'(\gamma, r)$ have discontinuities (obviously they may only be first order), we define them on discontinuity points up to functions $\underline{q}_2(\gamma, r)$ and $\bar{q}_1(\gamma, r)$ respectively, so that these functions are basis functions. And precisely when r' are the discontinuity points of $\underline{q}_2'$, or $\bar{q}_1'$, $r' \in \text{int } A$, then

$$\underline{q}_2(\gamma, r') = \{q \mid \underline{q}_2'(\gamma, r' - 0) \leq q \leq \underline{q}_2'(\gamma, r' + 0)\}$$

or

$$\bar{q}_1(\gamma, r') = \{q \mid \bar{q}_1'(\gamma, r' - 0) \leq q \leq \bar{q}_1'(\gamma, r' + 0)\},$$

where $q(\gamma, r' - 0)$ and $q(\gamma, r' + 0)$ are respectively left and right limits of the corresponding functions. We note that $\bar{q}_1(\gamma, r)$ and $\underline{q}_2(\gamma, r)$ are nondecreasing functions.

Lemma 2. If $L_\gamma \neq \phi$, then $\bar{q}_1(\gamma, r) \leq \underline{q}_2(\gamma, r)$ for all $r \in A$ and $L_\gamma = M_\gamma$ where

$$M_\gamma = \{x(r) \mid \bar{q}_1(\gamma, r) \leq x \leq \underline{q}_2(\gamma, r),\ x(r) \in L_\gamma\}.$$

The proof of the lemma is given in the appendix.

Let $M_\gamma \neq \phi$. We consider the function

$$\pi_\gamma^*(r) = \begin{cases} \underline{q}_2(\gamma, r^\ell), & \text{if} \quad r^\ell \leq r \leq a, \\ \bar{q}_1(\gamma, r), & \text{if} \quad a < r \leq r^h, \end{cases} \tag{9}$$

where $a = r^h$, if $\underline{q}_2(\gamma, r^\ell) \geq \bar{q}_1(\gamma, r^h)$, or a is a solution of the equation $\bar{q}_1(\gamma, a) = \underline{q}_2(\gamma, r^\ell)$, if $\underline{q}_2(\gamma, r^\ell) < \bar{q}_1(\gamma, r^h)$.

Substituting (9) in (7) and (8) we get the following theorem.

Theorem 2. The optimal functional mechanism μ^* is determined from the expression

$$\pi_{\gamma^*}^*(r) = \begin{cases} \underline{q}_2(\gamma^*, r^\ell), & \text{if} \quad r^\ell \leq r \leq a, \\ \bar{q}_1(\gamma^*, r), & \text{if} \quad a < r \leq r^h, \end{cases}$$

$$\sigma^*(x) = \begin{cases} 0 & \text{for } x < \pi_{\gamma^*}^*(r^\ell), \\ \int_{\pi_{\gamma^*}^*(r^\ell)}^{x} \zeta_r(t, \tilde{r}_{\gamma^*}(t))\,dt & \text{for } \pi_{\gamma^*}^*(r^\ell) \leq x \leq \pi_{\gamma^*}^*(r^h), \\ g & \text{for } x > \pi_{\gamma^*}^*(r^h), \end{cases}$$

where the efficacy of the optimal mechanism $\gamma^* = \gamma^*(g)$ is determined from (7) and $\tilde{r}_{\gamma^*}(t)$ is a function inverse to $\pi_{\gamma^*}(r)$.

Proof of the theorem is given in the appendix.

To illustrate these results we consider the following.

Example. Let $\Psi = cx - x^2/2r^2$, $\Psi_w = c^2r^2/2$, $\zeta = x^2/2r^2$ then (5) according to Lemma 1 may be put in the form $\beta_1(\gamma)r^2 \leq x \leq \beta_2(\gamma)r^2$, where $\beta_1(\gamma) = c(1 - \sqrt{1-\gamma})$, $\beta_2(\gamma) = c(1 + \sqrt{1-\gamma})$. The solution of problem (1) is equal to

$$\pi_{\gamma^*}^*(r) = \begin{cases} \beta_2(\gamma^*)(r^\ell)^2, & \text{if} \quad r^\ell \leq r \leq r^\ell\sqrt{\beta_2(\gamma^*)/\beta_1(\gamma^*)}, \\ \beta_1(\gamma^*)r^2, & \text{if} \quad r^\ell\sqrt{\beta_2(\gamma^*)/\beta_1(\gamma^*)} \leq r \leq r^h, \end{cases}$$

$$\sigma^*(x) = \beta_1(\gamma^*)x - \beta_1(\gamma^*)\beta_2(\gamma^*)(r^\ell)^2 \quad \text{for } \pi_{\gamma^*}^*(r^\ell) \leq x \leq \pi_{\gamma^*}^*(r^h),$$

where γ^* is defined as a solution of the equation

$$\beta_1^2(\gamma^*)(r^h)^2 - \beta_1(\gamma^*)\beta_2(\gamma^*)(r^\ell)^2 = g.$$

4. CONCLUSIONS

The model considered might be used, for example, to describe and analyze procedures for concluding agreements between customers and suppliers. In this model the customer is the center and the supplier the AE. The customer initiates the task $x = \pi(r)$ in the supplier by means of a bonus $\sigma(x)$. The resulting state allows estimation of the form of the stimulus function depending on the amount of information and the reform of the cost function.

One direction for future development of these results may be investigation of models in which there is a dependence on the central preference function on the stimulus functions, and a model system with several active elements.

APPENDIX

Proof of Theorem 1. We first show that π^{OC} is nondecreasing, i.e., from $r_1 < r_2$ it follows that $\pi^{OC}(r_1) \le \pi^{OC}(r_2)$. We assume the contrary, i.e., $\pi^{OC}(r_1) > \pi^{OC}(r_2)$. From condition (2) we write the inequality

$$\sigma(\pi^{OC}(r_1)) - \zeta(\pi^{OC}(r_1), r_1) \geqslant \sigma(\pi^{OC}(r_2)) - \zeta(\pi^{OC}(r_2), r_1),$$
$$\sigma(\pi^{OC}(r_2)) - \zeta(\pi^{OC}(r_2), r_2) \geqslant \sigma(\pi^{OC}(r_1)) - \zeta(\pi^{OC}(r_1), r_2).$$

Adding these equations we obtain

$$\zeta(x_1, r_1) - \zeta(x_1, r_2) \leqslant \zeta(x_2, r_1) - \zeta(x_2, r_2),$$

where $x_1 = \pi^{OC}(r_1)$, $x_2 = \pi^{OC}(r_2)$. From this inequality, the inequality $\dot{\zeta}_r(x_1, r) \ge \dot{\zeta}_r(x_2, r)$ for some $r \in [r_1, r_2]$, follows, contradicting for $x_1 > x_2$ the assumptions on the properties of the cost function, $\ddot{\zeta}_{xr}(x, r) < 0$. Thus, for the assumptions made above about the cost function, the procedure $\pi^{OC}(r)$ is nondecreasing, i.e., each $\pi^{OC}(r)$ belongs to L_γ for some $\gamma \ge 0$.

We now show that the solution of problem B gives the solution of problem (1). Thus let the mechanism $\mu^B = (\sigma(\pi^*), \pi^*)$, where $\pi^* = \pi^*_{\gamma^*}(r)$, be a solution of problem B. From (7) it follows that the mechanism $\mu^B \in M$, i.e., it is a possible solution of problem 1. We note that the performance index of the mechanisms μ^B is equal to $\gamma^B = \min_{r \in A}[\Psi(\pi^*_{\gamma^*}(r), r)/\Psi_w(r)] \ge \gamma^*$.

Let the mechanism $\mu^1 = (\tilde{\sigma}(\pi^1), \pi^1)$ be a solution of problem (1) and $\gamma^1 = K(\mu^1)$. We assume the contrary, i.e., that μ^B is not a solution of problem (1), implying $\gamma^1 > \gamma^B$.

Since $\pi^1(r)$ is nondecreasing, $\pi^1(r) \in L_{\gamma^1} \neq \phi$ and $\gamma^1 \le \gamma^M$. Since $L_{\gamma^B} \supset L_{\gamma^1}$ for $\gamma^1 > \gamma^*$, $0 \le \tilde{\sigma}(\pi^1) \le g$ and (6) is satisfied, then μ^B is not a solution of problem B, contradicting the initial assumptions. It follows from this contradiction that the mechanism μ^B is a solution of problem (1).

We note that $\tilde{\sigma}_\gamma = \tilde{\sigma}(\pi_\gamma(r^h))$ is an increasing function of the parameter γ. This follows from (6) and the properties of the set L_γ. And since $0 \le \tilde{\sigma}_\gamma \le g$, larger values of $\tilde{\sigma}_\gamma$ correspond to larger values of γ. Therefore the maximal value $\gamma = \gamma^*$ is either equal to γ^M or achieved for $\tilde{\sigma}_\gamma = g$. QED.

Proof of Lemma 1. Since $\Psi(x, r)$ is quasiconcave in x then $\Psi(x, r)/\Psi_w(r)$ is quasiconcave. A necessary and sufficient quasiconcavity condition [9] is convexity of the set $X(\gamma) = \{x \mid \Psi(x, r)/\Psi_w(r) \ge \gamma\}$ for any scalar γ. Therefore this set is a section, i.e., the set $X(\gamma)$ may be written as

$$q_1(\gamma, r) \leqslant x \leqslant q_2(\gamma, r), \tag{A.1}$$

where $q_1(\gamma, r)$ and $q_2(\gamma, r)$ for given γ and r are some numbers. We fix γ satisfying the lemma conditions and show that $q_1(\gamma, r)$ and $q_2(\gamma, r)$ are functions of r whose graphs are connected sets.

We note first that (5) describes a closed set of points (x, r). Indeed, let there be given a sequence of points $\{x_j, r_j\}$, satisfying (5) and let there exist a limit point (x, r) of this sequence. We show that this point also satisfies (5). We assume the contrary, i.e., that the point (x, r) does not satisfy (5), i.e., $\Psi(x, r) < \gamma\Psi_w(r)$, then because of the continuity of $\Psi(x, r)$ and $\Psi_w(r)$ there is a large enough number n such that $\Psi(x_n, r_n) < \gamma\Psi_w(r_n)$, but this contradicts the original assumptions, proving the closure.

We now consider one of the functions $q_1(\gamma, r)$ or $q_2(\gamma, r)$. We choose, for example $q_1(\gamma, r)$ and assume there exists a point $r' \in A$, at which the function $q_1(\gamma, r)$ is discontinuous, and therefore violates the connectivity of the graph. Thus without losing generality we assume that $q_1(\gamma, r' - 0) < q_1(\gamma, r' + 0)$. Because of the closure of the set defined by condition (5), the points $(q_1(\gamma, r' - 0), r')$ and $(q_1(\gamma, r' + 0), r')$, belong to this set, but then because of (A.1) all points (x, r'), such that $q_2(\gamma, r') \le x \le a_1(\gamma, r')$, belong to this set, therefore also the points (x, r'), such that $q_2(\gamma, r') \le q_1(\gamma, r' - 0) \le x \le q_1(\gamma, r' + 0)$, i.e., at the points r' the connectivity of the graph of the function $q_1(\gamma, r)$ is not violated. Similarly for $q_2(\gamma, r)$, proving the lemma.

Proof of Lemma 2. Let $L_\gamma \neq \phi$ and $\pi(r)$ be basis functions, $\pi(r) \in L_\gamma$. We show that

$$\bar{q}_1(\gamma, r) \leqslant \pi(r) \leqslant \underline{q}_2(\gamma, r). \tag{A.2}$$

We assume the contrary, i.e., for example $\bar{q}_1(\gamma, r') > \pi(r')$ for some r'. From the definition of $\bar{q}_1(\gamma, r)$ either $\bar{q}_1(\gamma, r') = q_1(\gamma, r')$ or there exists a point r'' such that $r'' < r'$ and $\bar{q}_1(\gamma, r') = q_1(\gamma, r'')$. In the first case it follows from $\bar{q}_1(\gamma, r') > \pi(r')$ that $q_1(\gamma, r) > \pi(r')$, which is impossible because $\pi(r) \in L_\gamma$. In the second case it follows from $\pi(r) \in L_\gamma$ that $\pi(r'') > q_1(\gamma, r'')$, but then $\pi(r'') > \pi(r')$ for $r'' < r'$. This contradicts the fact that $\pi(r)$ is a basis function. Therefore (A.2) holds. Because $\pi(r)$ is arbitrary in class L_γ, the second assertion of the lemma is true.

Proof of Theorem 2. We show that (10) and (11) define a solution of problem B. Then from Theorem 1, Theorem 2 will follow.

First we establish that the function π_γ^* determined by Eq. (9) for a given value of γ satisfies (6). Suppose there exists $\pi_\gamma(r) \in L_\gamma$, such that $\tilde{\sigma}(\pi_\gamma(r^h)) < \tilde{\sigma}(\pi_\gamma^*(r^h))$. We consider some function $\pi_\gamma(r) \in L_\gamma$. Suppose $\tilde{r}_\gamma(t)$ is the inverse of $\pi_\gamma(r)$, and $\tilde{r}_\gamma^*(t)$ is the inverse of $\pi_\gamma^*(r)$. We consider the equations

$$\Delta_\gamma = \tilde{\sigma}(\pi_\gamma(r^h)) - \tilde{\sigma}(\pi_\gamma^*(r^h)) = \int_{\pi_\gamma(r^\ell)}^{\pi_\gamma(r^h)} \dot{\zeta}_t(t, \tilde{r}_\gamma(t))\,dt - \int_{\pi_\gamma^*(r^\ell)}^{\pi_\gamma^*(r^h)} \dot{\zeta}_t(t, \tilde{r}_\gamma^*(t))\,dt = I_1 + \Delta' + I_2,$$

where

$$I_1 = \int_{\pi_\gamma(r^\ell)}^{\pi_\gamma^*(r^\ell)} \dot{\zeta}_t(t, \tilde{r}_\gamma(t))\,dt, \quad I_2 = \int_{\pi_\gamma^*(r^h)}^{\pi_\gamma(r^h)} \dot{\zeta}_t(t, \tilde{r}_\gamma(t))\,dt,$$

$$\Delta' = \int_{\pi_\gamma^*(r^\ell)}^{\pi_\gamma^*(r^h)} [\dot{\zeta}_t(t, \tilde{r}_\gamma(t)) - \dot{\zeta}_t(t, \tilde{r}_\gamma^*(t))]\,dt.$$

Clearly because $\pi_\gamma(r^\ell) \le \pi_\gamma^*(r^\ell)$ and $\pi_\gamma^*(r^h) \le \bar{q}_1(\gamma, (r^h) \le \pi_\gamma(r^h)$ then $I_1 \ge 0$, $I_2 \ge 0$. We now consider Δ'.

Since $\pi_\gamma(r^\ell) \le \pi_\gamma^*(r^\ell)$ and $\pi_\gamma(r^h) \ge \pi_\gamma^*(r^h)$, and also because of the connectivity of the graph of the function $\pi_\gamma(r)$, $\pi_\gamma^*(r)$ there exists a point $r' \in [r^\ell, r^h]$ for which $\pi_\gamma(r') = \pi_\gamma^*(r')$ and $\pi_\gamma(r) \le \pi_\gamma^*(r')$ for all $r \le r'$, and $\pi_\gamma(r) > \pi_\gamma^*(r')$ for all $r > r'$. From this it follows that $\tilde{r}_\gamma(t) \le \tilde{r}_\gamma^*(t)$ for all $t \in [\pi_\gamma^*(r^\ell), \pi_\gamma^*(r^h)]$. We prove this by contradiction, i.e., assume there exists $x \in [\pi_\gamma^*(r^\ell), \pi_\gamma^*(r^h)]$, such that $\tilde{r}_\gamma(x) > \tilde{r}_\gamma^*(x)$. We note that $x = \pi_\gamma(\tilde{r}_\gamma(x))$ and $x = \pi_\gamma^*(\tilde{r}_\gamma^*(x))$. Because $r' < \tilde{r}_\gamma$ and fixing of the point r' we have $\pi(\tilde{r}_\gamma) > \pi^*(\tilde{r}_\gamma)$. Therefore $\pi^*(\tilde{r}_\gamma^*) > \pi^*(\tilde{r}_\gamma)$, where $\tilde{r}_\gamma(x) > \tilde{r}_\gamma^*(x)$ from the assumptions. The equality we have obtained contradicts the monoticity of $\pi^*(r)$. Thus we have shown that $\tilde{r}_\gamma(t) \le \tilde{r}_\gamma^*(t)$. From this and the fact that $\ddot{\zeta}_{xr}(x, r) < 0$, it follows that $\Delta' \ge 0$. Therefore in turn it follows that $\Delta > 0$, i.e.,

$$\tilde{\sigma}(\pi_\gamma(r^h)) \geqslant \tilde{\sigma}(\pi_\gamma^*(r^h)).$$

This implies that the functions $\pi_\gamma^*(r)$ and $\tilde{\sigma}(\pi^*)$ are solutions of problem A. For a value γ^* of the parameter γ, determined from condition (7) these functions determine a solution of problem B. Then, as noted above, Theorem 2 follows from Theorem 1.

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