ELEMENTS OF THE THEORY OF OPTIMAL SYNTHESIS OF FUNCTIONING MECHANISMS OF TWO-LEVEL ACTIVE SYSTEMS

I. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR REGULAR FUNCTIONING MECHANISMS WITH COMPLETE INFORMATION AT THE CENTER

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The article derives necessary and sufficient optimality conditions for regular functioning mechanisms (i.e., mechanisms ensuring the fulfillment of planned targets) of a two-level active system on a set of mechanisms with fixed objective function and a fixed incentive system under conditions of complete information at the center. The results constitute a generalization of previously derived sufficient conditions.

1. The description of a "fanning-out" active system with independent state selection by the elements includes the following concepts [1]:

1) system state indexes $y = \{y_{ij} | j \in J_i,\ i \in I\}$, where J_i is the set of state components of element i, $I = \{i | i = 1, 2, \ldots, n\}$ is the set of all the elements in the system;

2) target indexes $x = \{x_{ij} | j \in J_i,\ i \in I\}$ in cases when all the components of state y have planned targets;

3) the set of feasible states of the system $Y \subseteq \prod_{i \in I} Y_i$, where Y_i is the set of feasible states of element i;

4) the set of feasible system targets $X \subseteq \prod_{i \in I} X_i$, where X_i is the set of feasible targets of element i;

5) the functioning mechanism of system Σ, which includes the objective function of the system Φ, the system of incentives for the elements $f = \{f_i | i \in I\}$ (where f_i is the objective function of element i), and a planning procedure π used to formulate the target x;

6) the set of locally optimal states of independent elements $P(f, x) = \{y \in Y | f_i(x_i, y_i) \geq f_i(x, z_i),\ z_i \in Y_i,\ i \in I\}$;

7) an efficiency criterion of the functioning mechanism $K(\Sigma) = \min \Phi(x, y)$ over $y \in R(\Sigma)$, where $R(\Sigma) = \{y | y = x, \text{ if } x \in P(f, x), \text{ else } y \in P(f, x)\}$ is the set of solutions of a game played by the elements, assuming friendliness and locally optimal behavior;

8) the set of feasible functioning mechanisms G_Σ.

In the theory of active systems, the optimal synthesis of functioning mechanisms involves determining a mechanism $\hat{\Sigma}$ which satisfies the additional constraints $\hat{\Sigma} \in G_\Sigma{}^a$ and ensures maximum functioning efficiency of the system on a given set of functioning mechanisms $G_\Sigma{}^g \subseteq G$ in the sense of the criterion $K(\Sigma)$ [1]:

$$K(\hat{\Sigma}) = \max_{\Sigma \in G_\Sigma^g} K(\Sigma), \quad \hat{\Sigma} \in G_\Sigma^g \cap G_\Sigma^a. \tag{1}$$

Since the functioning mechanism of a system includes several components, it is meaningful to consider the synthesis of one or two components, while keeping all the others fixed. A number of studies have dealt with optimal synthesis of planning procedures in a system with independent elements [1-7]. In problems of this type $G_\Sigma{}^g \subseteq G_\pi$, where G_π is the set of functioning mechanisms with a fixed objective function of the system

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Φ and a fixed incentive system of the elements f. Since on the set G_π the planning procedure is the only variable component of the functioning mechanism, we use for $K(\Sigma)$ and $R(\Sigma)$ the simpler notation $K(x)$ and $R(x)$, respectively, where x is the target generated by the planning procedure π_Σ. The following planning procedures were considered in detail:

$$\pi^{OP}:\Phi(x,x)\underset{x}{\to}\max,\quad x\in X\subseteq Y,$$

$$\pi^{OPP}:K(x)\underset{x}{\to}\max,\quad x\in X,$$

$$\pi^{OPC}:\Phi(x,x)\underset{x}{\to}\max,\quad x\in X\cap S,$$

where $S=\{x|\max f_i(x_i, y_i) = f_i(x_i, x_i)$ over $y_i\in Y_i,\ x_i\in Y_i,\ i\in I\}$ is the set of perfectly consistent realizable targets. The three planning procedures listed above are respectively termed optimal planning (OP), optimal planning with state prediction (OPP), and optimal perfectly consistent planning (OPC). The main property of perfectly consistent planning procedures is that each element is assigned a target in which it is actively interested. Therefore, assuming rational behavior of the elements, perfectly consistent plans will always be fulfilled. Functioning mechanisms ensuring target attainment are termed regular. We denote by $\tilde{G}_\Sigma$ the set of regular mechanisms. In this case the OPC procedure is a solution of problem (1) for $G_\Sigma^g \subseteq G_\pi$ and $G_\Sigma^a = \tilde{G}_\Sigma$.

Optimal synthesis of incentive systems satisfying additional constraints, such as the condition of limited penalties, were also considered [1].

In the forthcoming cycle of articles, we consider the optimal synthesis of planning procedures and incentive systems which maximize the system functioning efficiency and satisfy certain additional constraints. These constraints may include the requirement of unconditional attainment of the targets in all or in some of the components, the requirement of exceeding the targets, etc. The first requirement is the basis of the principle of perfect consistent planning [1]. The consistent planning principle also makes it possible to introduce a number of additional constraints. We will derive necessary and sufficient optimality conditions on various sets of functioning mechanisms. The results obtained for the case of complete information at the center are generalized to the case of partial information at the center regarding the possibilities and the interests of the subordinate elements.

In this article, we derive necessary and sufficient conditions for the optimality of regular functioning mechanisms on G_π. These results generalize the sufficient conditions previously derived in [1, 5-7]. In a certain sense they substantiate the principle of perfectly consistent planning for the case of complete information at the center.

2. Consider a system of independent elements with planned targets for all the state components of the elements (the technique which can be used to extend the results to the case of partial planning will be found in [1]). We assume that the system incurs a penalty if the state y deviates from the target state x:

$$\Phi(y,y)\geqslant\Phi(x,y) \tag{2}$$

on the set $X \times Y$. It is further assumed that the elements are friendly toward the central authority and select their states from the set of locally optimal states. We denote by X^{OPP} the set of targets generated by the OPP procedure, $X^{OPP} = \mathrm{Arg}\max K(x)$ over $x\in X$; Σ^{OPC} and Σ^{OPP} are functioning mechanisms with procedures OPC and OPP, respectively: $X(x) = \{z|z = x\}$, if $x\in R(x)$, else $z\in X$. Consider the following conditions:

1°. $K(\Sigma^{OPP})=K(\Sigma^{OPC})$;

2°. $X^{OPP}\cap S\neq\phi$;

3°. $\exists x\in X^{OPP}: R(x)\cap X\cap S\neq\phi$;

4°. $\exists x\in X^{OPP},\ z\in R(x)\cap X: \forall y\in Y,\ i\in I: f_i(z_i, z_i)\geqslant f_i(z_i, y_i)$;

5°. $\exists x\in X^{OPP},\ z\in X(x)\cap Y: \forall y, y'\in Y,\ i\in I$:

$$f_i(z_i,y_i)+f_i(x_i,y_i)\leqslant f_i(z_i,z_i)+f_i(x_i,z_i);$$

6°. $\exists x\in X^{OPP},\ z\in X(x)\cap Y: \forall y\in Y,\ \alpha,\beta\geqslant 0,\ i\in I$:

$$\alpha f_i(z_i,y_i)+\beta f_i(x_i,y_i)\leqslant\alpha f_i(z_i,z_i)+\beta f_i(x_i,z_i).$$

THEOREM 1. Conditions 1°-6° are equivalent.

The proof is given in the Appendix.

3. Let us discuss the results of the theorem. The nonemptiness of the intersection $X^{OPP}\cap S \neq \phi$ directly implies that the OPP targets include at least one perfectly consistent realizable target.

The requirements imposed on incentive systems are incorporated in conditions 1°-6° in implicit form. In particular, conditions 3°-6° include the set X^{OPP}, which is determined both by the incentive system of the elements f and by the objective function of the system Φ. It is therefore fairly difficult to apply the conditions 1°-6° in the analysis of incentive systems and especially in synthesis problems. The natural way out of this difficulty is by devising relatively simple constructive sufficient conditions in order to restrict as far as possible the set of optimal regular functioning mechanisms. Successively simplifying the conditions 3°-6°, we can derive a whole set of sufficient conditions.

In order to eliminate the dependence of conditions 3°-6° on the form of the objective function of the system Φ, we replace the requirement $\exists x \in X^{OPP}$ by a stronger requirement $\forall x \in X$. The resulting conditions, in general, are not necessary, but they depend only on the objective functions of the elements (although implicitly as before).

We represent the objective functions of the elements f_i, $i \in I$, in the form $f_i(x_i, y_i) = h_i(y_i) - \chi_i(x_i, y_i)$, where $h_i(y_i) = f_i(y_i, y_i)$, $\chi_i(x_i, y_i)$ is the penalty function for failure to achieve the target.

Conditions 3°-6° now can be rewritten in the form

3°a. $\forall x \in X : R(x) \cap X \cap S \neq \phi;$

4°a. $\forall x \in X : \exists z \in R(x) \cap X : \forall y \in Y,\ i \in I:$

$$\chi_i(z_i, y_i) \geqslant h_i(y_i) - h_i(z_i);$$

5°a. $\forall x \in X : \exists z \in X(x) \cap Y : \forall y, y' \in Y,\ i \in I:$

$$h_i(y_i) - h_i(y_i') - 2h_i(z_i) + \chi_i(x_i, z_i) \leqslant \chi_i(z_i, y_i) + \chi_i(x_i, y_i');$$

6°a. $\forall x \in X,\ z \in X \cap Y,\ y \in Y,\ i \in I:$

$$\Delta_i(y_i) \leqslant \chi_i(x_i, y_i) - \chi_i(x_i, z_i) \leqslant \chi_i(z_i, y_i),\ \chi_i(x_i, y_i) \geqslant 0;$$

where

$$\Delta_i(y_i) = \begin{cases} -\infty, & \text{if } y_i \in X_i \cap Y_i, \\ \max\limits_{i \in Y_i} h_i(y_i) - \max\limits_{z_i \in X_i \cap Y_i} h_i(z_i), & \text{if } y_i \notin X_i \cap Y_i. \end{cases}$$

Sufficiency of conditions 3°a, 4°a, and 5°a is obvious. Let us prove sufficiency of condition 6°a. Suppose that 6° does not hold, while 6°a is satisfied. Then $\forall x \in X^{OPP}$, $z \in X(x) \cap Y : \exists y \in Y$, $\alpha, \beta \geq 0$, $i \in I$:

$$\alpha f_i(z_i, y_i) + \beta f_i(x_i, y_i) > \alpha f_i(z_i, z_i) + \beta f_i(x_i, z_i). \tag{3}$$

Inequality (3) is a fortiori true if we replace $\forall x \in X^{OPP}$ with $\exists x \in X$. From the constraint $\forall x \in X$, $y \in Y$, $z \in X \cap Y$, $i \in I$: $\Delta_i(y_i) \leq \chi_i(x_i, y_i) - \chi_i(x_i, z_i)$ in condition 6°a, we have that $\forall x \in X : R(x) \cap X \neq \phi$. Take $z \in R(x) \cap X$, then $\forall y \in Y$, $i \in I : f_i(x_i, y_i) \leq f_i(x_i, z_i)$, and the strict inequality in (3) holds only if $\exists i \in I : \forall y \in Y$ we have the strict inequality $f_i(z_i, z_i) < f_i(z_i, y_i)$. If this is so, then inequality (3) also holds for $\alpha = -\beta \neq 0$. Comparing the inequality obtained from (3) with the inequality in 6°a, we obtain an obvious contradiction. Thus, 6°a implies 6°.

Further simplifying conditions 3°a, 4°a, and 6°a, we obtain the previous sufficient conditions of maximum consistency, "strong penalties," and strong consistency respectively [1]:

3°b. $S = \bigcup\limits_{x \in X} R(x) \subseteq X;$

4°b. $\forall x, y \in Y \subseteq X,\ i \in I:$

$$\chi_i(x_i, y_i) \geqslant h_i(y_i) - h_i(x_i);$$

6°b. $\forall x \in X,\ z, y \in Y \subseteq X,\ i \in I:$

$$0 \leqslant \chi_i(x_i, y_i) \leqslant \chi_i(z_i, y_i) + \chi_i(x_i, z_i).$$

4. The set of all optimal regular functioning mechanisms defined by conditions 1°-6° in general is wider the set of functioning mechanisms defined by any of the sufficient conditions. To prove this statement, consider a simple example. Take an incentive system which does not satisfy the strong consistency condition 6°b, but which nevertheless may be used to construct an optimal regular functioning mechanism.

Consider a simple two-level system consisting of the central authority and one subordinate element. The objective function of the system has the form $\Phi(x, y) = y$. The objective function of the element is

$$f(x, y) = h(y) - \chi(x, y),$$

where $h(y) = 2yc - y^2$ and

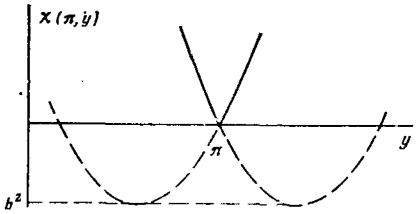

Fig. 1

$$\chi(x,y)=\begin{cases}(y-x-b)^2-b^2 & \text{for } y\leqslant x,\\ (y-x+b)^2-b^2 & \text{for } y\geqslant x.\end{cases}$$

To fix ideas, let $Y = X = [0, a]$ and $0 < c \leq a$. The penalty function χ is a strictly convex function which for these Y and X does not satisfy condition 6°b [1]. The penalty function is shown graphically in Fig. 1.

Application of the OPP procedure yields $X^{OPP} = a$. Seeing that the set of perfectly consistent realizable targets has the form $S = [\max(c - b, 0), \min(c + b, a)]$, we can apply condition 2° of the theorem to derive a condition of optimality of the regular functioning mechanism for the given incentive system: $b \geq a - c$.

APPENDIX

Proof of Theorem 1. If condition 1° implies condition 2°, we write 1° => 2°. The equivalence of the conditions for 1° => 2°, 3° => 2°, 5° => 3°, 6° => 3° is proved by contradiction.

1° => 2°. Let $X^{OPP}\cap S = \phi$. Since the mechanism Σ^{OPC} exists, we have $S\cap X \neq \phi$. From the definition of Σ^{OPP} we have

$$K(\Sigma^{OPP}) = \max_{x\in X} K(x) = K(x^1),$$

where $x^1 \in X^{OPP}$. Since $\forall x^2 \in (S\cap X)\setminus X^{OPP} = S\cap X$, we have the inequality $K(\Sigma^{OPP}) > K(x^2)$. Therefore,

$$K(\Sigma^{OPP}) > \max_{x\in S\cap X} K(x) = K(\Sigma^{OPC}),$$

which contradicts 1°.

2° => 1°. Let $x^1 \in X^{OPP}\cap S$. Then we can write the following chain of equalities:

$$K(\Sigma^{OPP}) = \max_{x\in X} K(x) = K(x^1) = \max_{x\in S\cap X} K(x) = K(\Sigma^{OPC}),$$

which proves the implication.

2° => 3°. Take an arbitrary $x \in X^{OPP}\cap S$. From the definition of the set S it follows that $x \in R(x)$ and so condition 3° is satisfied.

3° => 2°. Let $X^{OPP}\cap S = \phi$. Using this assumption and condition 3°, we can write that $\exists x^1 \in X^{OPP} : \exists x^2 \in R(x^1)\cap S\cap X$ and $x^2 \in X\setminus X^{OPP}$. Now we can write two chains of inequalities:

$$K(x^2) < K(\Sigma^{OPP}),$$
$$K(\Sigma^{OPP}) = K(x^1) = \min_{y\in R(x^1)} \Phi(x^1, y) \leqslant \Phi(x^1, x^2) \leqslant \Phi(x^2, x^2) = K(x^2).$$

Here we have used the property (2) of the objective function of the system. The resulting contradiction proves the implication.

3° => 4°. Since for some $x \in X^{OPP} : R(x)\cap X\cap S \neq \phi$, we have that $\exists z \in R(x)\cap X\cap S$. From the definition of the set S it follows that $\forall y \in Y : f_i(z_i, z_i) \geq f_i(z_i, y_i)$, which proves the implication.

4° => 3°. By 4°, $\forall x \in X^{OPP} : (R(x)\cap X\cap S = \phi$. Hence $\exists x \in X^{OPP} : \forall z \in R(x)\cap X : \exists y \in Y, i \in I : f_i(z_i, z_i) < f_i(z_i, y_i)$, which contradicts 4°.

4° => 5°. Since $\exists x \in X^{OPP}$, $z \in R(x)\cap X \subseteq Y\cap X(x) : \forall y' \in Y, i \in I$, we have $f_i(x_i, y_i') \leq f_i(x_i, z_i)$. Comparing this inequality with 4°, we obtain 5°.

5° => 3°. Then $\forall x \in X^{OPP} : R(x)\cap S\cap X = \phi$. In this case $\forall z \in X(x)\cap Y : z \notin R(x)\cap S$. This implies that either $z \notin R(x)$ or $z \notin S$ or both. In this case, these conditions lead to an inequality which contradicts the inequality in 5°. The contradiction establishes condition 3°.

The implications 4° => 6° and 6° => 3° are proved along the same lines as 4° => 5° and 5° => 3°, respectively. Q.E.D.

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