Automation and Remote Control, Vol. 56, No. 7, 1995

AN EFFICIENT ALGORITHM FOR SOLVING A SPECIAL CASE OF THE GENERALIZED PROBLEM OF STONES

V. N. Burkov[1], S. I. Dzyubko[1], and A. A. Yagupov[2]

UDC 519.854.2

An efficient method for solving a special case of the classic "problem of stones" is considered. The problem is the division of n different objects (stones) into m groups (piles) so that, wherever possible, the total volumes of all groups are equal. Consideration is given to the case where the volumes are ordered so that the volume of the jth object is described by a polynomial of degree α. For the case of $n \equiv 0 \pmod{2m^\alpha}$, an algorithm with computation time estimated as $O(n)$ is proposed and some generalizations of the problem are discussed.

INTRODUCTION

This work considers one of the classic problems of scheduling theory and discrete programming, the so-called "problem of stones," which has numerous applications to programming, operations research, and control [1, 2]. The algorithm proposed here can prove useful in optimal scheduling of identical units of equipment or of computing facilities, for packing repetitive lots of products of different volume into identical containers, as well as for solving any other problem which yields to it.

1. FORMULATION OF THE PROBLEM

In formal terms, the problem is as follows: given are n different objects (stones) and the set of their volumes $\{v_j\}$, $j = \overline{1, n}$, and it is needed to divide the entire totality of stones into m parts (piles) so that, wherever possible, the total volumes of all parts are equal. The complexity of the problem is due to its discreteness (the stones cannot be broken), that is, it is required to find the set of variables $\{x_{jk}\}$, $j = \overline{1, n}$ and $k = \overline{1, m}$, such that $x_{jk} = 1$ if the jth stone is placed into the kth pile, or $x_{jk} = 0$ otherwise. Now we have the following model:

$$\prod_j \sum_k x_{jk} = 1, \qquad \max_k \sum_j v_j x_{jk} \longrightarrow \min. \tag{1}$$

The above problem of discrete programming belongs to the class of computationally complex NP-complete programs [1, 2]. It is known [3] that for each particular problem a polynomial

$$f(x) = d_\alpha x^\alpha + d_{\alpha-1} x^{\alpha-1} + \ldots + d_0 \tag{2}$$

exists such that all volumes can be ordered so that

$$v_j = f(j), \quad j = \overline{1, n}; \tag{3}$$

here, the coefficients $d_\alpha, d_{\alpha-1}, \ldots, d_0$ can be any real numbers and α can be an integer.

The special case considered below differs from the general problem in that the following condition must be satisfied:

$$n \equiv 0 \pmod{2m^\alpha}. \tag{4}$$

---

[1]Institute of Control Sciences, Russian Academy of Sciences, Moscow. [2]RUSSINKO Co., Moscow. Translated from Avtomatika i Telemekhanika, No. 7, pp. 124–130, July, 1995. Original article submitted September 29, 1994.

2. DESCRIPTION OF THE METHOD OF SOLUTION

First, we consider the case where the condition $V_j = j^\alpha$ is satisfied and for this purpose introduce the function of distribution of volumes $\varphi(i,k)$, where i is the internal number of the stone in the pile and k is the number of the pile. In the case under consideration, each pile has the same number of stones, denoted by Q. Let the jth-in-order stone be the ith in the kth pile; then $\varphi(i,k) = j$.

The values of the functions can be determined explicitly for $\alpha = 1$ or inductively for $\alpha > 1$.

For $\alpha = 1$, we consider a sequence of any $2m$ stones. Then we have $Q = 2$, $V_j = j$, and the kth pile will have stones with numbers k and $2m - k + 1$. Therefore,

$$\varphi(1,k) = k,$$
$$\varphi(2,k) = 2m - k + 1.$$

We prove that, for this distribution, the total volumes of all piles are equal.

Indeed, according to the distribution, the total volume V_k of the kth pile is

$$V_k = \varphi(1,k) + \varphi(2,k);$$

then,

$$V_k - V_{k+1} = \sum_{i=1}^{2} \varphi(i,k) - \sum_{i=1}^{2} \varphi(i,k+1) = k + 2m - k + 1 - (k+1) - (2m-k) = 0, \tag{5}$$

and the total volumes of all parts are equal because k is arbitrary.

To determine the distribution function in the general case, we need the parameter u, which will appear in each addend in the left side as an additional addend. Then, it follows from (5) that

$$\sum_{i=1}^{2} [\varphi(i,k) + u] - \sum_{i=1}^{2} [\varphi(i,k+1) + u] = 0$$

for any value of u.

It follows from the last equality that the form of the distribution function is independent of the reference point and, therefore, for $n > 2m$ the problem can be solved independently for any sequence of $2m$ volumes, and the satisfaction of the condition (4) suffices for getting the precise solution.

Now we prove by induction the existence of such a function for any arbitrary α. To this end, we assume that for any positive integer degree α there exists a function similar to the function for the degree $\alpha = 1$, that is, that the following expression holds:

$$\sum_{i=1}^{Q} \Big\{ [\varphi(i,k) + u]^\alpha - [\varphi(i,k+1) + u]^\alpha \Big\} = 0, \tag{6}$$

where Q is the minimal number of stones in one pile required for α if (6) is satisfied. It follows from (6) that

$$\sum_{i=1}^{Q} \sum_{\beta=0}^{\alpha} C_\alpha^{\alpha-\beta} u^{\alpha-\beta} [\varphi^\beta(i,k) - \varphi^\beta(i,k+1)] = \sum_{\beta=0}^{\alpha} C_\alpha^{\alpha-\beta} u^{\alpha-\beta} \sum_{i=1}^{Q} [\varphi^\beta(i,k) - \varphi^\beta(i,k+1)] = 0.$$

Let now

$$f(\beta) = C_\alpha^{\alpha-\beta} \sum_{i=1}^{Q} [\varphi^\beta(i,k) - \varphi^\beta(i,k+1)].$$

We note that $f(0) = 0$. Then, with due regard for the last notation, we get that

$$\sum_{\beta=1}^{\alpha} u^{\alpha-\beta} f(\beta) = 0. \tag{7}$$

We note further that since by assumption the last equality is equal to zero for any u, it must also be equal to zero for $u = 1$, that is,

$$\sum_{\beta=1}^{\alpha} f(\beta) = 0 \tag{8}$$

is valid.

We prove that the next assertion follows from the assumption (7).

Assertion 1:

$$f(\beta) = 0, \quad \beta = \overline{1, \alpha}. \tag{9}$$

Proof. This assertion is proved by contradiction. Indeed, let there be β such that

$$f(\beta) \neq 0. \tag{10}$$

We note that, since the values of the functions $f(\beta)$ are the differences of some subsets of volumes, the inequality

$$|f(\beta)| \geqslant 2 \tag{11}$$

can always be satisfied by multiplying the volumes by the same sufficiently large number.

We note that the property (3) holds here for all volumes and the form of $\varphi^{\alpha}(i, k)$ is as before. Next, we note that if there is only one nonzero addend, (7) cannot be satisfied. Therefore, some addends in this case must be positive and some negative, that is, if A is the set of all nonzero addends, it follows that $|A| \geqslant 2$. Let $A_0 \subset A$ be a subset of addends for which $f(\beta) < 0$ and $A_p \subset A$ be a subset where $f(\beta) > 0$ for each element. Then, $A_p \cup A_0 = A$.

Proceeding from the assumption (10), the sum of the absolute values of all negative addends must be equal here to that of all positive addends, that is,

$$\sum_{\beta \in A_0} u^{\alpha-\beta} |f(\beta)| = \sum_{\beta \in A_p} u^{\alpha-\beta} |f(\beta)| \tag{12}$$

must hold true.

Since (9) must be satisfied for any u by assumption, it must also be valid for $u = \max\limits_{\beta} f^2(\beta)$. Now, let $f^2_{\max} = \max\limits_{\beta} f^2(\beta)$ and $u = f^2_{\max}$.

We note that the sum of the terms of the geometric progression S_α with denominator u equals [4]

$$S_\alpha = \sum_{\beta=0}^{\alpha} u^{\alpha-\beta} = \frac{1 - u^{\alpha+1}}{1 - u}$$

and, thus, we have

$$u^{\alpha+1} = 1 + S_\alpha(u - 1),$$

but then the inequality

$$u^{\alpha+1} > S_\alpha$$

holds for any $u \geqslant 2$. From the last inequality we get

$$f^{2\alpha-2}_{\max} > \sum_{l=1}^{2\alpha-3} f^{l}_{\max}.$$

We also note that

$$\sum_{\beta=2}^{\alpha} |f(\beta)| f^{2(\alpha-\beta)}_{\max} < \sum_{\beta=2}^{\alpha} f^{2\alpha-2\beta+1}_{\max}$$

and

$$\sum_{l=1}^{2\alpha-3} f_{\max}^{l} > \sum_{\beta=2}^{\alpha} f_{\max}^{2\alpha-2\beta+1},$$

because of all the addends in the left side only the odd ones are available in the right side. Therefore, $f_{\max}^{2\alpha-2} > \sum_{\beta=2}^{\alpha} |f(\beta)|\, f_{\max}^{2(\alpha-\beta)}$, but then

$$u^{\alpha-1}|f(1)| > \sum_{\beta=2}^{\alpha} u^{\alpha-\beta}|f(\beta)|.$$

This means that, on the strength of the assumption (10), one can always find a sufficiently large u such that one of the addends will be greater than the sum of the absolute values of the rest of the addends, but here the equality (12) cannot be satisfied; therefore, the assumption (10) is not true and Assertion 1 is valid, which is what we set out to prove.

Next we note that it follows from Assertion 1 that

$$f(\beta) = C_{\alpha}^{\alpha-\beta} \sum_{i=1}^{Q} \left[\varphi^{\beta}(i,k) - \varphi^{\beta}(i,k+1)\right] = 0, \quad \beta = \overline{1,\alpha}, \tag{13}$$

but $C_{\alpha}^{\alpha-\beta} > 0$ for $\beta = \overline{1,\alpha}$; therefore the totality of the equalities (13) can be valid only if the following takes place:

$$\sum_{i=1}^{Q} \left[\varphi^{\beta}(i,k) - \varphi^{\beta}(i,k+1)\right] = 0, \quad \forall\beta = \overline{1,\alpha}. \tag{14}$$

Now we consider (6) for the degree $\alpha+1$:

$$\sum_{i=1}^{Q} \left\{ \left[\varphi(i,k) + u\right]^{\alpha+1} - \left[\varphi(i,k+1) + u\right]^{\alpha+1} \right\} =$$

$$= \sum_{\beta=0}^{\alpha} C_{\alpha+1}^{\alpha-\beta+1} u^{\alpha-\beta+1} \sum_{i=1}^{Q} \left[\varphi^{\beta}(i,k) - \varphi^{\beta}(i,k+1)\right] + \sum_{i=1}^{Q} \left[\varphi^{\alpha+1}(i,k) - \varphi^{\alpha+1}(i,k+1)\right],$$

and since (14) holds, the last expression is equal to

$$\sum_{i=1}^{Q} \left[\varphi^{\alpha+1}(i,k) - \varphi^{\alpha+1}(i,k+1)\right],$$

which is independent of u. Therefore, if for the degree α there exists a function $\varphi^{\alpha}(i,k)$ such that all differences between the total volumes can be constructed to be equal to zero, its application to the problem of degree $\alpha+1$ gives the same set of differences for any sets of numbers that are necessary for solving the problem of degree α.

If now

$$\Delta(k,\alpha+1) = \sum_{i=1}^{Q} \left[\varphi^{\alpha+1}(i,k) - \varphi^{\alpha+1}(i,k+1)\right],$$

we get

$$\begin{aligned}
V_1 - V_2 &= \Delta(1,\alpha+1),\\
V_2 - V_3 &= \Delta(2,\alpha+1),\\
&\cdots\cdots\cdots\cdots\\
V_k - V_{k+1} &= \Delta(k,\alpha+1),\\
&\cdots\cdots\cdots\cdots\\
V_{m-1} - V_m &= \Delta(m-1,\alpha+1)
\end{aligned}$$

or

$$V_1 = V_m + \sum_{j=1}^{m-1} \Delta(j, \alpha+1),$$
$$V_2 = V_m + \sum_{j=2}^{m-1} \Delta(j, \alpha+1),$$
$$\dots\dots\dots\dots\dots\dots$$
$$V_k = V_m + \sum_{j=k}^{m-1} \Delta(j, \alpha+1),$$
$$\dots\dots\dots\dots\dots\dots$$
$$V_{m-1} = V_m + \Delta(m-1, \alpha+1).$$

Then, by enumerating m times the numbers of parts so that each time the part number is incremented by 1 if $k < m$ or by assuming it to be equal to 1 if $k = m$, we determine the total volume of the kth part

$$V_k = \sum_{q=k+1}^{m} \left[V_m + \sum_{j=q}^{m-1} \Delta(j, \alpha+1)\right] + \sum_{q=1}^{k} \left[V_m + \sum_{j=q}^{m-1} \Delta(j, \alpha+1)\right] =$$
$$= \sum_{q=1}^{m} \left[V_m + \sum_{j=q}^{m-1} \Delta(j, \alpha+1)\right] = mV_m + \sum_{q=1}^{m} \sum_{j=q}^{m-1} \Delta(j, \alpha+1),$$

which is independent of k. Therefore, the total volumes of all parts are $V_1 = V_2 = \ldots = V_m$ and all differences are zero.

Thus, having the function $\varphi^{\alpha}(i, k)$ for the degree α and using it for m-fold solution of the problem of degree $\alpha + 1$ with cyclic change of the part numbers we construct the functions $\varphi^{\alpha+1}(i, k)$ for solution of the problem of degree $\alpha + 1$, which is what we set out to prove.

The minimal number of stones required for solution of the problem of degree $\alpha + 1$ is mQ, but since $2m$ stones are required for $\alpha = 1$, it follows that $2m^{\alpha}$ stones are required for the degree α, that is, the condition (4) must be satisfied.

3. DESCRIPTION OF THE ALGORITHM

It will be assumed in the description of the algorithm that the condition (12) is satisfied, the set of volumes $\{v_i\}$ is determined from (2), and $\{A_j\}$ and $\{P_j\}$ are the sets of auxiliary variables required for determination of the indices.

Algorithm 1

1. Input α, m, n, $\{v_i\}$, $V = 0$, $k = \overline{1, m}$, $i = \overline{1, n}$, $S = 0$.
2. $i = k = 0$, $q = r = 1$, $\beta = \alpha$, $P_j = A_j = 0$, $j = \overline{1, 2m^{\alpha}}$.
3. $j = 0$.
4. $j = j + 1$.
5. $i = i + 1$, $k = k + 1$.
6. If $P_j < m$, then $P_j = P_j + 1$; otherwise $P_j = 1$.
7. $A_i = P_j$.
8. $l = S + P_j$, $h = S + 2m - P_j + 1$.
9. $x_{lk} = 1$, $x_{hk} = 1$.
10. $V_k = V_k + v_l + v_h$.
11. If $k = m$, then $k = 0$, $S = S + 2m$.
12. If $j < q$, then proceed to Step 4.
13. $r = r + 1$.
14. If $r \leqslant m$, then proceed to Step 3.
15. $k = 0$, $\alpha = \alpha - 1$, $q = mq$, $r = 2$, $P_j = A_j$, $j = \overline{1, q}$.
16. If $\alpha > 0$, then proceed to Step 3.
17. $\alpha = \beta$, $n = n - 2m^{\alpha}$.
18. If $n > 0$, then proceed to Step 2.
19. Output $\{x_{ij}\}$, $\{V_j\}$, $i = \overline{1, n}$, $j = \overline{1, m}$.
20. End.

We note that upon solving a problem of degree α by Algorithm 1, all problems of lower degrees are solved automatically. Thus, the problems of all degrees are solved independently. We also note that the distribution is independent of the polynomial constants because all volumes are incremented here by the same number. Therefore, if all volumes are ordered before applying Algorithm 1 so that the condition (3) is satisfied, then it guarantees the global optimal solution and the computation time is estimated as $O(n)$ because the algorithm handles each volume only once.

Therefore, the following theorem is valid.

THEOREM 1. For the generalized problem of stones (1) and the conditions (3) and (4), one can always distribute the stones using Algorithm 1 with time estimate $O(n)$ so that the total volumes of the piles will be equal

$$V_1 = V_2 = \ldots = V_m.$$

Example. Let $v_i = i^2 - 10i + 25$, $m = 2$, and the minimal $n = 2m^2 = 8$.
After distributing the first four volumes using Algorithm 1, we have

$$x_{11} = 1, \quad x_{41} = 1, \quad x_{22} = 1, \quad x_{32} = 1,$$
$$V_1 = v_1 + v_4 = 16 + 1 = 17,$$
$$V_2 = v_2 + v_3 = 9 + 4 = 13.$$

Finally, after distributing the remaining volumes we have

$$x_{61} = 1, \quad x_{71} = 1, \quad x_{52} = 1, \quad x_{82} = 1,$$
$$V_1 = 17 + v_6 + v_7 = 17 + 1 + 4 = 22,$$
$$V_2 = 13 + v_5 + v_8 = 13 + 0 + 9 = 22.$$

4. DISCUSSION

The above method is applicable to other problems. Let us discuss one of them. Let a polynomial of degree α be defined over the interval $[a, b]$. It is required to divide the interval into m domains so that the areas bounded by the polynomial be equal for all domains. To solve this problem, we decompose the interval into $2m^\alpha$ equal segments. The area bounded by the polynomial can be shown to be a polynomial function of the number of the segment of degree α. The problem is solvable by the above algorithm. Interestingly, the solution is independent of the form of the polynomial only if its degree does not exceed α.

We consider another generalization. Let it be required to distribute the stones between k piles so that, wherever possible, the relations of pile volumes be close to the given relations $a_1 : a_2 : \ldots : a_k$. To solve the problem, we find the least common multiple q of all numbers a_i, determine

$$m = \sum_{i=1}^{k} \frac{a_i}{q},$$

and solve the problem of division into equal piles. If the number of stones is $m \equiv 0 \pmod{2m^\alpha}$, it is solved by the above algorithm. This and other problems will be discussed in more detail in the paper to follow.

REFERENCES

1. M. Gary and D. Johnson, *Computers and Intractability*, Wiley (1978).
2. E. G. Coffman, Jr. (ed.), *Computer and Job-Shop Scheduling Theory*, Wiley (1976).
3. N. S. Bakhvalov and N. P. Zhidkov, *Numerical Methods* [in Russian], Nauka, Moscow (1987).
4. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill (1961).