MATHEMATICAL SIMULATION OF COMPETITION MECHANISMS

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A competition model of an economic system is discussed in which the elements are described by convex functions of production costs. The problem of developing an optimal competition mechanism is stated and solved.

1. Introduction

Socialist competition has been and remains a strong motive for improving the efficiency of production and intensifying economics. The investigation of socialist-competition problems by using the methods of control theory, the development on this basis of recommendations regarding the perfection of forms and methods for organizing competition are today's urgent tasks. At the April 1985 Plenary Session of the Central Committee of the Communist Party of the Soviet Union Premier M. S. Gorbachev remarked that it is time to set about the improvement of organization structures in management, to simplify the staff, and to enhance its efficiency [1]. In recent years a number of articles have appeared in which an attempt is made to simulate mathematically the mechanisms of socialist competition from the vantage points of the theory for active systems [2, 3]. The businesslike games "Competition" and "Quality" [4] have been developed to study experimentally the mechanisms of competition. In this article a simple model of an economic system is considered where the elements are described by convex functions of the production costs. The problem of developing an optimal competition mechanism is stated and solved.

2. Description of the Model. Statement of the Problem

We will discuss a two-level active system comprising a center and n active elements. We will introduce the designations: y_i for the state of the i-th element (the number of a delivered product), $z_i = \varphi_i(y_i)$ for the function of the production costs, which describes expenditures of the element to deliver the product in the quantity y_i, $i = \overline{1, n}$. Next we will assume that $\varphi_i(y_i)$ is an increasing convex (rigorously) function $y_i > 0$. We will consider the following competition mechanism. Let $y_{i_1} \geq y_{i_2} \geq \ldots \geq y_{i_n}$ be deliveries of the product with elements in decreasing order. The first m elements $i_1, i_2, \ldots, i_m$ are declared winners of the competition and their prize is $\lambda_y y_i$, $i = i_1, i_2, \ldots, i_m$ $\forall m \in \{1, n\}$.

For the remaining (n − m) elements the prize is μy_i, $i = i_{m+1}, \ldots, i_n$, where $\mu < \lambda_i$, $i = \overline{1, n}$. With allowance for expenditures of special-purpose functions of winner elements of the competition we will write

$$f_i(\lambda_i, y_i) = \lambda_i y_i - \varphi_i(y_i), \tag{1}$$

and for the remaining elements

$$f_i(\mu, y_i) = \mu y_i - \varphi_i(y_i). \tag{2}$$

A study of this model is reduced to a study of the characteristics for solving a corresponding game of n players.

In order to evaluate the effectiveness of a competition mechanism for a given $\{\lambda_i\}$ and μ it is necessary to define hypotheses concerning the behavior of the players (elements) and the solution of a game $y^* = \{y_i^*\}$ determined by these hypotheses as a certain equilibrium condition of the system. Let Q be a set of winners in solving a game and R be a set of the remaining elements. For the elements $i \in R$ the choice of the conditions $y_i^* = w_i$ is determined by the natural specification of maximizing Eq. (2), i.e.,

$$\mu w_i - \varphi_i(w_i) = \max_{y_i} [\mu y_i - \varphi_i(y_i)] = \Delta_i. \tag{3}$$

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For the elements $i \in Q$ (the winners) the behavioral hypothesis is more complicated. First of all we will establish the value of $v_i(\lambda_i)$ as the maximal root of the equation

$$\lambda_i v_i - \varphi_i(v_i) = \Delta_i, \quad i=\overline{1, n}. \tag{4}$$

The meaningful sense of the quantity $v_i(\lambda_i)$ is that only for delivery of a product $y_i < v_i(\lambda_i)$ is an element "useful as a winner." We now note that if for some $i \in Q$ we have

$$y_i^* < \max_{i \in R} v_i(\lambda_i) = v_j(\lambda_j), \tag{5}$$

then an element j is useful, after having chosen the condition $y_i^* < y_j < v_j$, as a winner. In this sense the state y^*, for which Eq. (5) is satisfied if only for one i, is not in equilibrium. On the other hand, if we should have

$$y_i^* \geqslant \max_{i \in R} v_i = v(R), \quad i \in Q, \tag{6}$$

$$\lambda_i y_i^* - \varphi_i(y_i^*) = \max_{y_i \geqslant v(R)} [\lambda_i y_i - \varphi_i(y_i)] \geqslant \Delta_i, \tag{7}$$

then the winner elements, first of all, need not fear that they will be "overrun" and, secondly, they are useful as winners. We note that if Eq.(6) is an equality, an element j for a delivery $y_j = v_j$ can be a winner (although its special-purpose function in this case is not changed). Similarly, when Eq. (7) is an equality and $y_i^* > w_i$, the element i can reduce delivery to the quantity w_i, having yielded the victory to another element, and its special-purpose function also is not changed. To avoid unnecessary complications during an investigation we shall adopt the hypothesis of "inertia of the elements" for which the elements do not alter strategy (the value of a delivery) if a change causes no increase of the special-purpose functions. It is easy to see that a maximum of $\lambda_i y_i - \varphi_i(y_i)$ is achieved for the condition of Eq. (6) when $y_i^* = \max(v(R), w_i(\lambda_i)) = u_i(R, \lambda_i)$, where $w_i(\lambda_i) = \operatorname{Arg}\max_{y_i} (\lambda_i y_i - \varphi_i(y_i))$. Finally, if the elements are split into two subsets m and n − m of elements, correspondingly, which satisfy Eqs. (6) and (7), then the condition

$$y_i^* = \begin{cases} u_i(R, \lambda_i), & i \in Q, \\ w_i, & i \in R \end{cases} \tag{8}$$

will be described as a solution of a competitive game. This definition of the solution for a competitive game was proposed in [4] and then employed to investigate the effectiveness of a number of simple competition mechanisms in [2]. It was shown that it can be regarded as a Π-solution.

There then arises the question of the existence and uniqueness of such solutions in the model under consideration. Let the elements be arranged in decreasing order of v_i, i.e., $v_{i1} \geq v_{i2} \geq \ldots \geq v_{in}$.

<u>THEOREM.</u> The solution of a competitive game will have the form

$$y_{i_k}^* = \begin{cases} \max(v_{i_{m+1}}, w_{i_k}(\lambda_{i_k})), & k = 1, 2, \ldots, m; \\ w_{i_k}, & k = m+1, \ldots, n. \end{cases} \tag{9}$$

The number of solutions $N(m) = C_{p(m)}^{m-q(m)}$, where p(m) is the number of values of v_i, which is equal to w_{im}, and q(m) is the number of values of v_i, which is less than w_{im}.

<u>Proof</u>. We will show that for $Q = \{i_k : k \leq m\}$, $R = \{i_k : k > m\}$ a solution of the game will have the form of Eq. (9). We will have

$$y_i^* = \begin{cases} w_i, & i \in R \\ \max(w_i(\lambda_i), \; v_{i_{m+1}}), & i \in Q. \end{cases}$$

The fulfillment of the conditions per Eq. (6) is obvious. Furthermore, we will have for $i \in Q$ in the case where $w_i(\lambda_i) \leq v_{im+1} \leq v_i$

$$\lambda_i v_{i_{m+1}} - \varphi_i(v_{i_{m+1}}) \geqslant \lambda_i v_i - \varphi_i(v_i) = \Delta_i.$$

In the case where $w_i(\lambda_i) > v_{im+1}$ we will have

$$\lambda_i w_i(\lambda_i) - \varphi_i(w_i(\lambda_i)) = \Delta_i(\lambda_i) \geqslant \Delta_i.$$

Thus the conditions of Eqs. (6) and (7) are satisfied, and in Eq. (9) there is a solution of the game. Now let there exist a solution in which $v_{ik} < v_{im}$ and $i_k \in Q$. In this case, $y^*_{i_k} \geqslant v(R) = \max_{i\in R} v_i \geqslant v_{i_m} > v_{i_k}$ which contradicts the equilibrium condition $y_{ik}^* \leq v_{ik}$. A solution is nonunique when $v_{im+1} = v_{im}$. The quantity $m - q(m)$ is equal to the number of winners with the values $v_i = v_{im}$, and $p(m)$ is the number of elements with the values $v_i = v_{im}$. The number of different subsets Q is obviously equal to the number of combinations from $p(m)$ up to $(m - q(m))$, i.e., $C_{p(m)}^{m-q(m)}$. The theorem is proved.

COROLLARY. If $\varphi_i(y_i) = \varphi(y_i)$, $\lambda_i = \lambda$, then $N(m) = C_n^m$. Thus a solution for a competitive game always exists with this model. Let there be given the quantity μ and the restrictions P for the total delivery of a product, i.e., $\sum_i y_i^* = P$. We will pose the problem of constructing an optimal competitive mechanism. Let G be a set of pairs (Q, R) that determine the solutions of a game in the form of Eq. (9).

Problem. To determine m and $\{\lambda_i\}$ such that

$$\sum_{i\in Q} \lambda_i y_i^* + \sum_{i\in R} \mu y_i^* \xrightarrow[(Q,R)\in G]{} \min$$

with the condition

$$\sum_i y_i^* = P.$$

Interestingly, the problem indicates a minimization requirement on the means for stimulating the elements with a given restriction on delivery of a product.

3. Case of Identical Elements

Let $\varphi_i(y_i) = \varphi(y_i)$ for all i. Let the elements be numbered according to decreasing v_i, i.e., $v_1 \geq v_2 \geq \ldots \geq v_n$ and, correspondingly, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Since for all $i \in Q$ the delivery of the product $y_i^* = v_{m+1}$, it then follows from the condition for minimizing the means of stimulation that $\lambda_i = \lambda_{m+1}$ for all $i = 1, m$. We will denote $\lambda_{m+1} = \lambda$, and $v_{m+1} = v$. The problem is then reduced to the minimization of

$$S = m\lambda v + (n-m)\mu w \tag{10}$$

for the restrictions

$$mv + (n-m)w = P. \tag{11}$$

From Eq. (11) we will have $v = (P - nw)/m + w = A/m + w$, where $A = P - nw > 0$, because $W < P/n$. From the condition of Eq. (5), $\lambda v - \varphi(v) = \Delta$ we will obtain

$$\lambda = \frac{\Delta + \varphi(v)}{v} = \frac{\mu w - \varphi(w) + \varphi(v)}{v}. \tag{12}$$

By substituting Eq. (12) into Eq. (10), we will finally have

$$S = \frac{A[\varphi(v) - \varphi(w)]}{v - w} + n\mu w.$$

As is well known, for convex functions φ the expression

$$\frac{\varphi(v) - \varphi(w)}{v - w}$$

is an increasing function $v > m$. Therefore, the minimum of S corresponds to minimal v, and this means maximal $m = n - 1$. We will obtain an important conclusion.

THEOREM 1. For convex functions of production costs in the case of identical elements the optimal number of winners is equal to n − 1. Interestingly, the result obtained corresponds to the familiar principle of competition "not to be last," when it is assumed that there is only one loser who has the worst result.

Thus the minimal value of the stimulating means is

$$S_{\min}(w)=\left[\varphi\left(\frac{P-w}{n-1}\right)-\varphi(w)\right](n-1)+nw\varphi'(w). \tag{13}$$

4. Exponential Functions of Production Costs

Let us consider in greater detail functions for production costs in the form $\varphi(y) = ky^{\alpha}$, $\alpha > 1$. In this case,

$$S_{\min}(w)=k\left\{\left[\left(\frac{P-w}{n-1}\right)^{\alpha}-w^{\alpha}\right](n-1)+n\alpha w^{\alpha}\right\}.$$

We will find the optimal value of w [so $\mu = \varphi'(w)$].

After simple computations we will have

$$w_{\text{opt}}=\frac{P}{1+(n-1)(\alpha n-n+1)^{\frac{1}{\alpha-1}}}<\frac{P}{n}.$$

We will investigate the dependence of w_{opt} on α and n. It is quite clear that w_{opt} is a decreasing function of n and an increasing function of α, where

$$\lim_{\alpha\to1} w_{\text{opt}}(\alpha)=\frac{P}{1+(n-1)e^{n}},$$
$$\lim_{\alpha\to\infty} w_{\text{opt}}(\alpha)=P/n.$$

It is interesting to compare the mechanism of competition with the mechanism of stimulation when the special-purpose functions of the elements are equal to $\mu y - \varphi(y)$ and the delivery of a product is determined by the condition $\varphi'(y) = \mu$. Obviously when y = P/n the value of the stimulating means is equal to

$$S_{\text{st}}=P\varphi'\left(\frac{P}{n}\right)=k\alpha\left(\frac{P}{n}\right)^{\alpha}n.$$

In the case of a competitive mechanism where $w = w_{\text{opt}}$ we will have

$$S_{\text{com}}=kP^{\alpha}\frac{1+(\alpha-1)n}{[1+(n-1)(\alpha n-n+1)^{\frac{1}{\alpha-1}}]^{\alpha-1}}.$$

We will determine the effectiveness of the competitive mechanism by the ratio

$$\rho(\alpha,n)=\frac{\alpha\{1+(n-1)[(\alpha-1)n+1]^{\frac{1}{\alpha-1}}\}^{\alpha-1}}{n^{\alpha-1}[1+(\alpha-1)n]}.$$

When n is large, $\rho(\alpha, n) \approx \alpha$, i.e., a competitive mechanism for large n is α times more effective than the usual stimulation. To evaluate the effectiveness with small n we note that $\rho(\alpha, n)$ is an increasing function of n, i.e., the effectiveness of competitive mechanisms increases as the number of competitors rises (the massive competition principle). For the minimal number n = 2 of elements we will have

$$\rho(\alpha)=\frac{\alpha[1+(2\alpha-1)^{\frac{1}{\alpha-1}}]^{\alpha-1}}{2^{\alpha-1}[2\alpha-1]}.$$

5. General Case

Let us consider the general case where there are k groups of elements such that within each group the elements are described by the same function of production costs $\varphi_k(y)$. In this case, of course, the competition is organized within each group. There then arises the problem of distributing a given P among the groups so that the required number of products would be delivered with minimal means of encouragement. In a formal statement it is required to find P_k and w_k so that the quantity

$$\sum_k \left\{ \left[\varphi_k \left(\frac{P_k - w_k}{n_k - 1} \right) - \varphi_k(w_k) \right] (n_k - 1) + n_k w_k \varphi_k'(w_k) \right\}$$

would be minimal with the condition

$$\sum_k P_k = P. \tag{14}$$

If $\varphi_k(y) = r_k y^{\alpha_k}$, then the problem comprises the minimization of

$$S = \sum_k r_k P_k^{\alpha_k} \frac{1 + (\alpha_k - 1) n_k}{\{1 + (n_k - 1)[(\alpha_k - 1) n_k + 1]^{\frac{1}{\alpha_k - 1}}\}^{\alpha_k - 1}}.$$

with the condition of Eq. (14). If n_k is sufficiently large for all k, then

$$S \simeq \sum_k r_k \left(\frac{P_k}{n_k} \right)^{\alpha_k} n_k$$

and the optimal solution satisfies the condition

$$\frac{P_k^*}{n_k} \sim \delta_k = r_k^{-\frac{1}{\alpha_k - 1}},$$

i.e., the planned assignment for one element is smaller, the larger the coefficient r_k in the function of production costs.

6. Conclusions

The results that have been obtained permit a number of qualitative conclusions to be drawn about the characteristics of optimal competitive mechanisms. Thus, the conclusion concerning the organization of competition within similar groups of elements with a maximal number of winners reflects the well-known Leninist principles of congruence (identity of conditions) of contenders and mass character of participation in the competition as important conditions for its effectiveness.

We note that these results permit another interesting interpretation as a competition for "stressed plans." For this purpose it is sufficient to take the quantity y_i as a counter-plan of an element, and $\varphi_i(y_i)$ as the expenditures to fulfill this plan. Finally, it is not complicated to generalize the results even in the case of vector states (plans) y_i. As a rule, in this case the tally of summed competitions is made by complex evaluations results for the activity $Q(y_i)$. Assuming that $K_i = Q(y_i)$ for the aggregated state of an element (or aggregated plan), and defining the cost function $\Phi_i(K_i)$ as the optimal solution of the task

$$\Phi_i(K_i) = \min \varphi_i(y_i)$$

with the condition $Q(y_i) = K_i$, $y_i \in Y_i$, where Y_i is a set of possible states (plans), we will obtain the model considered above with scalar elements.

The validity of the theoretical conclusions that have been obtained depends to a large extent on the hypothesis adopted for the behavior of the elements. Currently, experiments are being conducted at the Institute for Control Problems with the businesslike game "Competition" in order to check the bases for the hypothesis that has been adopted.

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