Efficient mechanism for resource allocation with quadratic payments and its realization via an iterative bargaining process

Korgin N*

** V.A. Trapeznikov Institute of Control Sciences,*
Moscow Institute of Physics and Technology
65 Profsoyuznaya, Moscow 117997, Russia
(Tel: +7495-335-60-37; e-mail: nkorgin@ipu.ru).
This work is supported by RFBR, project № 12-07-31244.

Abstract: The problem of Pareto-efficient resource allocation among rational agents is considered. The mechanism that implements efficient allocation as Nash equilibrium in case when utility is transferable among agents is offered. The approach to solution of allotment problems as multicriteria public choice problems lies in the basis of this mechanism, that allows to implement Groves-Ledyard mechanism, which was initially designated to the solution of public good problems. It is shown, that there is exist the only Nash equilibrium in a game among agents induced by the mechanism developed. For the case when utility functions are private information of agents, it is shown, that efficient allocation may be realized via an iterative bargaining process based on this mechanism, if agents behave according to Cournot dynamics. Possibility to reduce agent's messages space to a scalar one in iterative bargaining process is demonstrated. It is also shown that mechanism developed may be inconsistent for some nontrivial agent's behavior – there exist some game solutions which can be reached via iterative bargaining process but are not Nash equilibrium and don't yield efficient resource allocation.

Keywords: Control in organizational systems, Game theory, Resource allocation, Mechanism Design, Nash Implementation.

1. INTRODUCTION

We consider the problem of resource allocation – limited amount $R \in \mathbb{R}^1_+$ of infinitely divisible good should be allotted among finite number of agents from the society $N = \{1,...,n\}$. Each agent $i \in N$ has an utility function $u_i(\bullet): \mathbb{R}^1_+ \to \mathbb{R}$ that is drawn from some set of possible utility functions U_i. Let us denote the set of *feasible allocation* as

$$A = \{x = (x_1,...,x_n) : \sum_{i \in N} x_i \le R, \, x \in \mathbb{R}^n_+\},$$

the set of possible utilities profiles as

$$U = \{u = (u_1(\bullet),...,u_n(\bullet)) : u_i(\bullet) \in U_i, i \in N\}.$$

The goal is to find such *allocation mapping* $g(\bullet): U \to A$, that is *efficient* in sense that it maximizes the total utility of all agents for any utility profile $u \in U$:

$$g(u) \in Arg\max_{x \in A} \sum_{i \in N} u_i(x_i).$$

Even if for some set of utility profiles U such mapping does exist, it may be not *incentive compatible* (see Maskin (1985) for example). That is mean that given utility profile $u \in U$ there may exist an agent $k \in N$ and a utility profile $\tilde{u} = (\tilde{u}_k, u_{-k})$ such that

$$u_k(g_k(\tilde{u})) > u_k(g_k(u)),$$

where u_{-k} - the profile of preferences of all agents except k and $u = (u_k, u_{-k})$, $g_k(u)$ - the amount of good allotted to agent k given profile u.

This is the issue of Mechanism Design – is it possible to find such *mechanism* $\rho = < S, \pi, t >$ where $S = \times_{i \in N} S_i$ - a space of agent's actions, $\pi(\bullet): S \to A$ - some procedure that maps agent actions onto set of feasible allocation and $t(\bullet): S \to \mathbb{R}^n$ - transfers of utilities, that will allow to realise efficient allocation in case when $g(\bullet)$ is not incentive compatible.

We consider following set of utility profiles $\widehat{U}$:

1. utility function of each agent is concave and C^2;
2. $\forall u \in \widehat{U}$ it is not efficient to allocate all the good available to one agent - that is $\forall \{i, j\} \in N^2$

$$\frac{\partial u_i}{\partial x_i}(0) \ge \frac{\partial u_j}{\partial x_j}(R);$$

3. there exist possibility to transfer utility among agents.

We will provide the mechanism, that $\forall u \in \widehat{U}$ yields efficient allocation as the only *Nash equilibrium* in game $\Gamma(\rho) = < N, S, \varphi_{u,\pi} >$, where $\varphi_{u,\pi} = \{\varphi_1, ..., \varphi_n\}$ - profile of preferences of agents determined by their utility profile $u \in U$ and procedure $\pi(\bullet)$:

$$\varphi_i(s) = u_i(\pi(s)) - t_i(s),\ i \in N .$$

That is mean that $\forall u \in U\ \exists! s^*(u) \in S$:

1. $\forall i \in N$, $\forall \tilde{s}_i \in S_i\ \varphi_i(s^*(u)) > \varphi_i(\tilde{s}_i, s^*_{-i}(u))$;

2. $\pi(s^*(u)) = g(u)$.

Moreover, mechanism may be treated as efficient one when

$$\sum_{i \in N} \varphi_i(s^*(u)) = \sum_{i \in N} u_i(g(u)) .$$

That leads to very important property of mechanism with transfers – *balanced* transfers in equilibrium:

$$\sum_{i=1}^{n} t_i(s^*(u)) = 0 .$$

The problem considered has only one solution, when agent's utilities are not transferable and there is no additional information about agent's utilities – the *uniform* allocation, see for example Novikov (2013).

With transferable utility solutions for similar problems were developed by various authors. There are two main approaches to solution of the problem stated. The first one is *strategy-proof direct* mechanisms – when each agent is asked about his utility function and the best (*dominant*) strategy is to report information truthfully – so called Vickery-Groves-Clark (VGS) mechanisms. The drawbacks of this approach are following (see, for example, Maskin (1985) for general case, Yang S., Hajek B. (2005) for the case of problem considered):

1. It is not possible to balance transfers, that results in

$$\sum_{i \in N} \varphi_i(s^*(u)) < \sum_{i \in N} u_i(g(u))$$

2. Message space of such mechanism are quite complex – each agent should report its utility function, that results in nontrivial practical realisation of such mechanism.

Nash implementation approach allows to construct so called *indirect* mechanisms - agents may be not asked about their utility but about something else. Basar and Maheswaran (2003) developed class of mechanism with *proportional allocation rule*, which was later generalized by Yang and Hajek (2005) and Johari and Tsitsiklis (2009) in different ways. These mechanisms yield efficient allocation as the only Nash equilibrium, and have "smallest" message space – each agent is asked about his desirable share of the amount of good available.

But transfers are only *asymptotically balanced* – for any $P \in (0; \lambda)$ there exist transfers such that

$$\sum_{i=1}^{n} t_i(s^*(u)) = P ,$$

where λ - Lagrange multiplier for initial optimisation problem. But they didn't offer solution for $P = 0$.

Another close approach – *cost sharing* mechanisms that are developed by Moulin (see, for example Moulin (2010)). It also has small message space – each agent announce only amount of good, that he would like to receive. But this mechanism is also inapplicable if cost of good to be paid by whole society is zero.

In this paper following approach is utilized. Resource allocation problem is treated as multidimensional *public choice problem* – the vector of good's allocation among agents is treated as *public good.* This approach turned out to be fruitful for solution of resource allocation problem with out transferable utility – it allows do extend class of strategy-proof mechanism, see Burkov, Iskakov and Korgin (2010) and Korgin (2012)

For the case when each agent knows only its own utility function, we provide conditions when $s^*(u)$ may be achieved via *iterative bargaining process*:

$$x(\tau) = \pi(s(\tau - 1)),\ \varphi_i(\tau) = u_i(x(\tau)) - t(s(\tau)),$$

where $s(\tau) = (s_1(\tau), ..., s_n(\tau))$ - agents' messages at iteration $\tau \geq 1$.

2. DESCRIPTION OF THE MECHANISM

The mechanism $\rho = < S, \pi, t >$ that solves problem stated above is following. $\forall i \in N$

$$S_i = \{s_i \in \mathbb{R}^n : \sum_{j \in N} s_{ji} \leq R\} ,$$

$$x_i(s) = \frac{1}{n} \sum_{j=1}^{n} s_{ij}, \quad (1)$$

where s_{ij} is message of agent j about what amount of good should be allocated to agent i.

$$t_i(s) = p_i(s) - \frac{\alpha}{n} \sum_{j=1}^{n} p_j(s), \quad (2)$$

where

$$p_i(s) = \beta \sum_{j=1}^{n} (x_j(s) - s_{ji})^2 .$$

We will call parameters $\beta \geq 0$ - *penalty strictness,* $\alpha \in [0,1]$ - *balancing coefficient.* If $\alpha = 1$ then transfers are always balanced - $\forall s \in S$

$$\sum_{i=1}^{n} t_i(s) = 0$$

and mechanism may be treated as"Groves-Ledyard quadratic government" – mechanism, that was previously applied to public good problem (see Groves and Ledyard (1977), Arifovic and Ledyard (2011)).

For this mechanism the following statement is correct:

Proposition 1. $\forall u \in \widehat{U}$, $\forall \beta \geq 0$, $\forall \alpha \in [0,1]$ there is $\exists ! s^*(u) \in S$:

1. which is Nash equilibrium in game $\Gamma(\rho) = < N, S, \varphi_{u,\pi} >$;
2. $x(s^*(u)) = \arg\max_{x \in A} \sum_{i \in N} u_i(x_i)$.

In case of $\alpha = 1$ the mechanism offered maximizes the total utility of all agents. If $\alpha < 1$, then it yields efficient allocation of resource, but total utility of agents is less then the possible maximum:

$$\sum_{i=1}^{n} \varphi_i(s) < \max_{x \in A} \sum_{i=1}^{n} u_i(x_i)$$

Given $u \in \widehat{U}$ let us denote $s^*(\beta, \alpha)$ - solution of game depending on parameters of the mechanism.

Lemma 1. $\forall \beta \geq 0$, $\forall \alpha \in [0,1]$ $s^*(\beta, \alpha) = s^*(\tilde{\beta}, 0)$, where

$$\tilde{\beta} = \beta \frac{n - 1 - \alpha}{n - 1}.$$

That is why it is possible to balance transfers only in case when there are more then two agents.

Following equations describes dependencies between effective allocation and solution of the game. $\forall \{i, j\} \in N$:

$$x_i(s^*) = s^*_{ii} - \frac{\Delta}{n},$$

$$x_j(s^*) = s^*_{ji} + \frac{\Delta}{n(n-1)},$$

$$p_i(s^*) = \tilde{\beta} \frac{\Delta^2}{n(n-1)},$$

$$t_i(s^*) = \tilde{\beta}(1 - \alpha) \frac{\Delta^2}{n(n-1)},$$

where

$$\Delta = \sum_{i \in N} s^*_{ii} - R.$$

It is clearly seen from this equation, that if $\alpha = 1$, then there are no real transfers for any agent in equilibrium. Exact value of Δ depends from profile u - it should be derived as solution of the following equation:

$$\sum_{i \in N} u_i'^{-1}(2\tilde{\beta}\Delta) = R,$$

where $u_i'^{-1}(\bullet)$ is the inverse function to $u_i'(\bullet)$.

Another important property of this mechanism is that $\forall i \in N$ and $\forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j$ the *best response strategy* $br_i(s_{-i}) \in S_i$ (that gives maximum profit to agent, given that the profile of others' messages is s_{-i}) satisfies the following equations:

$$u_i'\left(\frac{1}{n}(br_{ii}(s_{-i}) + \sum_{k \in N \setminus \{i\}} s_{ik})\right) = \frac{2\tilde{\beta}}{n}\left[(n-1)br_{ii}(s_{-i}) - \sum_{k \in N \setminus \{i\}} s_{ik}\right] \quad (3)$$

and $\forall j \in N \setminus \{i\}$

$$(n-1)br_{ii}(s_{-i}) - \sum_{k \in N \setminus \{i\}} s_{ik} = (n-1)\left[\sum_{k \in N \setminus \{i\}} s_{jk} - (n-1)br_{ji}(s_{-i})\right]. \quad (4)$$

3. ITERATIVE BARGAINING PROCESS

One of the essential problems of Nash implementation is that Nash equilibrium may be not achieved by agents, if they are not fully informed about parameters of game they play – particularly if some agents don't have information about utility functions of others. That is why quite actual question is whether equilibrium may be reached via some learning process – when agent may acquire all the information about the game via interaction with each other (see Healy (2006)).

We consider the situation, when each agent initially knows only its own utility function, total number of agents, amount of the good available and mechanism.

Iterative bargaining process $I\rho$ under consideration is following. At initial stage (fist iteration) each agent announces amount of good he would like to receive - $s_{ii}(1) \in [0, R]$. If $\sum_{i \in N} s_{ii}(1) \leq R$, then $\forall i \in N$, and $\forall j \in N \setminus \{i\}$ we place $s_{ji}(1) = s_{ii}(1)$. If $\sum_{i \in N} s_{ii}(1) > R$ then $\forall i \in N$, $\forall j \in N \setminus \{i\}$ $s_{ji}(1) = (R - s_{ii}(1)) / (n - 1)$.

Amount of good that is offered to each agent and transfers are calculated according to (1) and (2) with $s = s(1)$. At each iteration $\tau > 1$ each agent can announce any feasible message $s_i(\tau) \in S_i$. Amount of good that is offered at iteration τ to each agent and transfers are calculated according to (1) and (2) with $s = s(\tau)$.

The bargaining process stops at step T if $\left\| s[T-1]-s[T] \right\| \le \varepsilon$. The final allotment and transfer are calculated according to (1) and (2) with $s=s(T)$.

We call agents behaviour *Cournot dynamics*, if at each iteration each agent chooses his messages as best response on messages of other agents at previous iteration:

$s_i(\tau)=br_i(s_{-i}(\tau-1))$,

where $br_i(s_{-i}(\tau-1))$ is derived from (3) and (4). In fact, at each step each agent tries to maximize his profit with assumptions that other agents will not change their messages.

For such iterative process following statement is correct

Proposition 2. Given $u\in\widehat{U}$ $s^*(u)$ is reachable in $I\rho$ if agents behaviour is Cournot dynamics and $\exists C\in\mathbb{R}$ such that $\forall i\in N$

$$\max_{i\in N}(-u_i''(x_i(s))\le C.$$

The iterative bargaining process $I\rho$ may be modified in order to reduce amount of information, that agents should announce at each iteration. $\forall\tau>1$ each agent is asked only about what share of good he would like to receive – $s_{ii}(\tau)$, $i\in N$. Then his messages about how much good should receive other agents are defined according to (4): $\forall j\in N\setminus\{i\}$ $s_{ji}(\tau)=br_{ji}(s_{-i}(\tau-1))$ with assumption that $br_{ii}(s_{-i}(\tau-1))=s_{ii}(\tau)$.

Modified iterative bargaining process also allows to reach $s^*(u)$ some $u\in\widehat{U}$ under conditions of proposition 2 taking in account that in order to perform Cournot dynamics agent should use (3) to define $s_{ii}(\tau)=br_{ii}(s_{-i}(\tau-1))$.

4. DIFFERENT BEHAVIOUR MODELS

Cournot dynamics is one of "simplest" in sense that agents behaviour model may be much more complicated – see for example Arifovic and Ledyard (2011). Following example illustrates situation, when such dynamics turns out to be «irrational» (while remaining to be rational at each iteration). Let us consider the modified iterative bargaining process.

There are 3 agents with utility functions $u_i=\sqrt{r_i+x_i}$ where $r=\{1;9;25\}$ - profile of initial endowment of agents. Amount of good to be divided among them is $R=115$.

Efficient allocation is $x=\{49,41,25\}$. In efficient allocation, profit of each agent is $\approx 7,07$.

Let us consider situation when at first iteration profile of agents' messages is $s(0)=\{115,115,115\}$. According to (1) and (2) $x(0)=\{115/3,115/3,115/3\}$ and $t(0)=\{0,0,0\}$. That is why $\forall i\in N$ $f_i(s(0))=u_i(x(0))$

For agent 3 $f_i(s(0))\approx 7,96$.

For mechanism with $\alpha=1$ and $\beta=10^{-3}$ it takes 7 iterations to reach allocation that is quite close to the efficient one, if agents behave themselves according to Cournot dynamics (see Fig. 1 and Fig. 2)

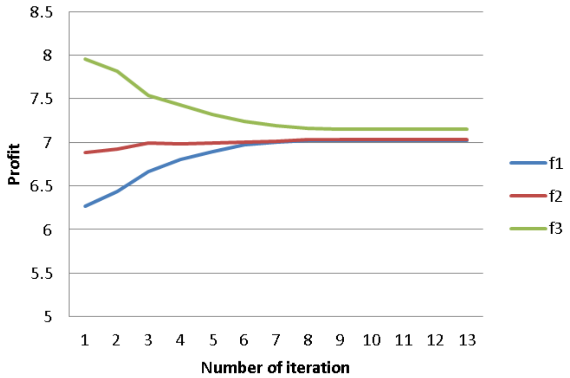

Fig. 1. Agents' profit under Cournot dynamics.

From Fig. 1 it is quite obvious, that at each iteration profit of agent 3 decreases. Moreover it is always less than expected under assumption that other agents will not change their messages. It turns out that best response strategy is not *secure strategy* (see Iskakov (2008)) for agent 3 at any iteration of bargaining process.

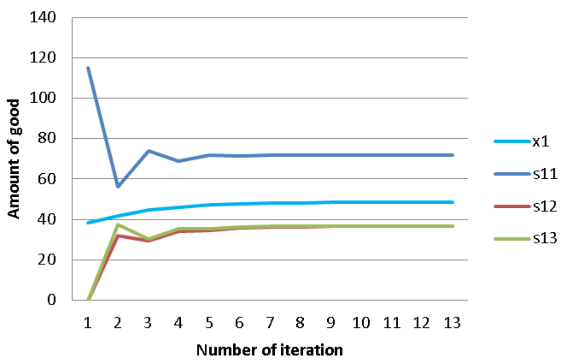

Fig. 2. Amount of good that is offered to agent 1 and messages about it from all agents under Cournot dynamics.

Figs 3 and 4 illustrates situation when agent 3 doesn't change his message $s_{33}=R$. Agents 1 and 2 acts according to their best responses till iteration 10. At second iteration agent 3 loses significantly, while agents 1 and 2 increase their profit . But from iteration 3 till 10 profit of agents 1 and 2 is significantly smaller than in initial allocation of at iteration 1.

At iteration 11 agent 2 changes his message to initial one - $s_{22}(11)=R$. While loosing profit at this iteration, from iteration 12 and further agent 2 receives more profit than at iterations where he picked best response strategy.

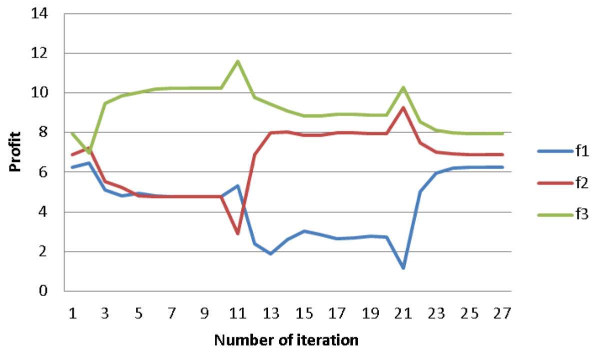

Fig. 3. Agents' profit – agent 3 doesn't follow Cournot dynamics.

From iteration 11 only agent 1 takes best response strategy. But at iteration 20 amount of good he receives and his best response message don't change significantly and his profit is only $\approx 2,73$. At iteration 21 agent 1 switch his message from best response for messages of agents 2 and 3 to $s_{11} = R$. And keeps this message for rest of iterations. From iteration 27 messages and profits of all agents are nearly the same as at first iteration.

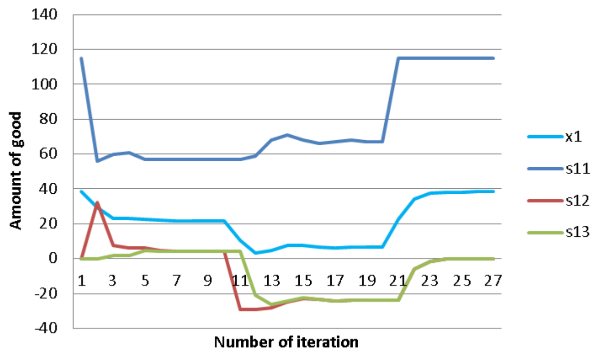

Fig. 4. Amount of good that is offered to agent 1 and messages about it from all agents when agent 3 doesn't follow Cournot dynamics.

At all iterations except 2, profit of agent 3 is higher than in effective allocation of good. Total utility of all agents is less then in effective allocation at all iterations.

6. CONCLUSIONS

Mechanism suggested in this paper has following advantages

1. It yield's efficient resource allocation as the only Nash equilibrium of game among agents.
2. Due to balanced transfers it does not reduce total utility of society.
3. In situation when utility functions are private information of the agents efficient allocation is achievable via iterative bargaining process based on mechanism offered.
4. Agent's message space in iterative bargaining process may be reduce to scalar.

The last of these points suggests a promising application of the mechanism developed - as a distributed optimization algorithm in style of *alternating direction method of multipliers (ADMM)*, see for example Boyd, Parikh and Chu (2011). This application should be explored further.

But the example provided makes it clear that even if mechanism induces the game with the only Nash equilibrium, it does not guarantee, that there will be other solutions of the game, that don't yield efficient allocation. Further explorations should be conducted in order to understand how it will perform under different assumptions about agents' behaviour, including possibility of cooperation among them. Equilibrium in secure strategies is one prospective concepts to deal with due to fact that mechanism design's technique for this concept is not developed yet while latest analysis of different classical models, tightly connected with recourse allocation problem shown that besides from Nash equilibrium (or even in case of its absence) there may exist an set of equilibria in secure strategies – see Iskakov and Iskakov (2012).

REFERENCES

Arifovic J., Ledyard J. O., (2011). A behavioral model for mechanism design: Individual evolutionary learning. *Journal of Economic Behavior and Organization*, 78, pp. 375–395.

Basar T., Maheswaran R. T., (2003). Nash equilibrium and decentralized negotiation in auctioning divisble resrouces. *Group Decision and Negotiation*, 12(5), pp. 361-395.

Boyd S., Parikh N., Chu E., (2011) Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers // *Foundations and Trends in Machine Learning*, Vol.3, №1, pp. 1-122.

Burkov V.N, Iskakov M.B., Korgin N.A., (2010). Application of generalized median voter schemes to designing strategy-proof mechanisms of multicriteria active expertise. *Automation and Remote Control*, 2010 Vol. 71, Num. 8, pp. 1681-1694 (original Russian text - Problemy Upravleniya, 2008, No. 4, pp. 38–47.)

Groves T., Ledyard J. O., (1977). Optimal allocation of public goods: A solution to the 'freerider' problem. *Econometrica* 45, pp. 783–809.

Iskakov M.B., (2008). Equilibrium in safety strategies and equilibriums in objections and counterobjections in noncooperative games, *Automation and Remote Control*, Vol 69, Issue 2, pp. 278-298.

Iskakov M. B., Iskakov A.B., (2012). Equilibrium in secure strategies, *Core discussion papers*, 2012/61, 35p.

Korgin N. A., (2012). Representation of sequential allotment rule as generalized median voter scheme with tie-breaking rule // *SSCW2012*. URL: https://editorialexpress.com/cgi-bin/conference/download.cgi?db_name=SSCW2012&paper_id=106

Healy P., (2006). Learning dynamics for mechanism design: an experimental comparison of public goods mechanisms. *Journal of Economic Theory*, 129, pp. 114–149.

Jain R., Walrand J., (2010). "An efficient Nash-implementation mechanism for divisible resource allocation", *Automatica*, vol. 46, no. 8, pp.1276 -1283 2010

Johari R., Tsitsiklis J. N., (2009). Efficiency of scalar-parameterized mechanisms. *Operations Research*, Vol. 57, N 4, pp. 823–839.

Maskin E., (1985). The Theory of Nash Equilibrium: A Survey. In L. Hurwicz, D. Schmeidler, and H. Sonnenschein, *Social Goals and Social Organization*, Cambridge: Cambridge University Press, pp.173-204.

Moulin H., (2010). An efficient and almost budget-balanced cost sharing method. *Games and Econom. Behavior*, Vol 70 Issue 1, pp. 107–131.

Novikov D.A., (2013). Theory of Control in Organizations. - *N.Y.: Nova Science Publishing*, 341 p.

Yang S., Hajek B., (2005). Revenue and stability of a mechanism for efficient allocation of a divisible good, *mimeo. Urbana Champaign: University of Illinois*, 35 p.