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SCALARIZATION OF THE CONSTRUCTION OF THE SLATER-OPTIMAL SOLUTION SET

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Multicriteria (vector) optimization is studied as a problem of constructing a set of x-consistent plans in an active system model. The construction of a set of Slater-optimal (semieffective) points in multicriteria optimization is reduced to optimization by a scalar criterion, which is a new convolution of the criterion set. The relationship of this convolution with the Karlin convolution is studied.

1. INTRODUCTION

A vital trend in operations research is multicriteria optimization. A basic tool in operations research is the reduction of one problem to another. Along with the representation of the technical apparatus for solving certain problems in terms of the solutions of other problem, such an approach yields additional information about these problems and new aspects in their interpretation.

In this paper, the interpretation of multicriteria optimization in terms of x-consistent planning is used to study a new criterial convolution, which is helpful in scalarizing the construction of a set of Slater-optimal solutions. For convex multicriteria optimization problems, the new convolution is shown to be related to the Karlin convolution.

2. MAIN RESULTS

Let us study the multicriteria optimization problem

$$f_i(y) \to \min, \quad y \in Y, \quad i \in I = \{1, 2, \dots, n\}. \tag{1}$$

In the sequel, without any special mention, we assume that all maxima and minima are attained. For this, it is sufficient to require that the functions be continuous and the sets be compact. If no additional information is available about the priorities of criteria, this problem is reduced to constructing a set of effective (Pareto-optimal) or semieffective (Slater-optimal) points

$$P = \{y \in Y \mid \forall z \in Y \;\; f_i(z) \leqslant f_i(y) \; \forall i \in I \Rightarrow f_i(z) = f_i(y)\},$$
$$S = \{y \in Y \mid \forall z \in Y \; \exists i \in I : f_i(z) \geqslant f_i(y)\}. \tag{2}$$

A large number of papers deal with the construction of these sets. The Karlin and Germeier techniques are the classical construction methods [1–3].

The Relationship between multicriteria optimization and optimal x-consistent planning for the base model of active systems [6–9] is studied in [4, 5]. The base model of a two-level active system with complete information consists of a center and n active elements. For our purposes, it suffices to study a system with one active element. The state of the system is determined by the state of the active element $y \in Y$, where Y is the set of all possible states of the active element. The center assigns a desirable state $x \in Y$ to the element, whereas the element chooses its state $y \in Y$ to match its interests best. The interests of the element are described by the stimulation function $f(x, y)$; so the decision that the element makes consists in solving the optimization problem

$$f(x, y) \to \max, \quad y \in Y. \tag{3}$$

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The center must adopt a plan $x \in Y$ with regard for its interests and the behavior of the element in the form of an aim function $\Phi(x, y)$. Different planning procedures arise, depending on the requirements that the center must take into account while making a decision. The most well known and deeply studied is the x-consistent planning, which is formulated as follows:

$$\Phi(x,x) \to \max, \quad x \in Y, \quad x \in X^{[x]}(f), \tag{4}$$
$$X^{[x]}(f) = \{x \in Y \mid f(x,y) \leqslant f(x,x) \ \forall y \in Y\}, \tag{5}$$

where $X^{[x]}(f)$ is a set of x-consistent plans of the element that are, by virtue of (5), advantageous in implementation, i.e., in choosing the state $y \in Y$ equal to x.

In [4, 5], multicriteria optimization is shown to be equivalent, in a certain sense, to the x-consistent planning. In other words, if a function of two variables is defined by the expression

$$f(x,y) = \min_{i \in I}(f_i(x) - f_i(y)),$$

we obtain the set-theoretic equality

$$X^{[x]}(f) = S.$$

A key role in the theory of active systems is played by the gain function of the active element

$$\varphi(x) = \max_{y \in Y} f(x,y), \tag{6}$$

which is the value of the objective function of the optimization problem (3) of the active element for the plane x. Furthermore,

$$x \in X^{[x]}(f) \iff \varphi(x) = f(x,x).$$

Using the gain function (6), for the multicriteria optimization problem (1), let us introduce the convolution

$$\varphi(x) = \max_{y \in Y}\left\{\min_{i \in I}(f_i(x) - f_i(y))\right\}. \tag{7}$$

The following lemma holds:

LEMMA 1. *A point $x \in Y$ is semieffective if and only if $\varphi(x) = 0$.*

The relation $f(x,x) \equiv 0$ implies that $\varphi(x) \geqslant 0$ for any x. Hence,

$$S = \operatorname{Arg}\min_{x \in Y} \varphi(x)$$

and (2) is equivalent to the set of semieffective points

$$S = \operatorname{Arg}\min_{x \in Y}\max_{y \in Y}\min_{i \in I}(f_i(x) - f_i(y)).$$

Therefore, the vector optimization problem (in the sense of constructing a set of semieffective points) is reduced to the scalar optimization problem

$$\varphi(y) \to \min, \quad y \in Y.$$

Example 1. Let us consider the multicriteria optimization problem

$$f_1(y) = (y+1)^2 \to \min,$$
$$f_2(y) = (y-1)^2 \to \min,$$
$$y \in R = (-\infty, +\infty).$$

Convolution (7) for this problem is

$$\varphi(x) = \max_{y \in R}\min(f_1(x) - f_1(y), f_2(x) - f_2(y)) = \max_{y \in R}\min((x-y)(x+y-2), (x-y)(x+y+2))$$
$$= \max_{y \in R}(x^2 - y^2 - 2|y - x|).$$

The superdifferential of the subextremal function, which is concave in y, may contain a zero only at the points 1, -1, and x. Therefore,

$$\varphi(x)=\max(x^2-1-2|1-x|,x^2-1-2|1+x|,0)=\begin{cases}0, & -1\leqslant x\leqslant 1,\\ (x+1)^2, & x\leqslant -1,\\ (x-1)^2, & x\geqslant 1.\end{cases}$$

Example 2. Let us consider the two-dimensional multicriteria optimization problem

$$f_1(y)=(y_1-1)^2+y_2^2\to\min,$$
$$f_2(y)=y_1^2+(y_2-1)^2\to\min,$$
$$y=(y_1,y_2)\in R^2.$$

Convolution (7) for this problem is computed in the same way as for the previous example and is equal to

$$\varphi(x)=\begin{cases}(x_1+x_2-1)^2/2, & |x_1-x_2|\leqslant 1,\\ (x_1-1)^2+x_2^2, & x_1-x_2\geqslant 1,\\ x_1^2+(x_2-1)^2, & x_1-x_2\leqslant -1.\end{cases}$$

Clearly, $\varphi(x)=0$ on the interval $[(0,1),(1,0)]$.

These examples show that the computation of convolution (7) is not easier than the construction of the set of semieffective points. However, if Y is a convex compact and $f_i(y)$ are convex continuous functions, we can use the following method to compute convolution (7). Let us note that if a constant equal to the minimum of a function is subtracted from the function, the minimum of the new function thus derived is zero and the point of minimum remains unchanged. Since every point $x\in S(f)$, by virtue of the Karlin lemma [1], can be determined as the solution of the optimization problem

$$\sum_{i\in I}\lambda_i f_i(y)\to\min,\quad y\in Y,$$

for certain nonnegative x-dependent $\lambda_i:\sum\lambda_i=1$, the convolution

$$\phi(x)=\min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\left(\sum_{i\in I}\lambda_i f_i(x)-\min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y)\right)\tag{8}$$

is self-evident. It is a simple matter to demonstrate

LEMMA 2. *Let Y be a convex compact set and let the functions $f_i(y)$ be convex and continuous. Then convolution* (8) *is such that*

$$\phi(x)\geqslant 0,\ \phi(x)=0\iff x\in S(f).$$

Hence convolutions (7) and (8) are similar and, under certain conditions, identical. Thus, we have

THEOREM 1. *Let Y be a convex compact set and let $f_i(y)$ be convex continuous functions. Then convolutions* (7) *and* (8) *are equal to*

$$\phi(x)=\varphi(x).$$

Let us note the following simple corollary:

COROLLARY 1. *Let the objective functions $f_i(y)$ be defined as $\mathrm{dist}^2(y,y_i)$, where $i\in I=\{1,2,\dots,n\}$, $y\in Y=R^m$ is an m-dimensional Euclidean space, $y_i\in Y$ are certain fixed points, and $\mathrm{dist}(x,y)$ is a function of the distance between two points. Then their convolution* (7) *is $\varphi(x)=\mathrm{dist}^2(x,A)$, where the set $A=\mathrm{conv}(y_1,y_2,\dots,y_n)$ is the convex shell of the point set $\{y_1,y_2,\dots,y_n\}$.*

Theorem 1 defines a relationship between convolution (7) and the Karlin convolution.

Example 3. The example below shows that convolution (7) for a one-dimensional case may be nonconvex even if the functions $f_i(y)$ are convex. Let

$$f_1(y) = |y|, \quad f_2(y) = |y-1|/2.$$

Then

$$\begin{aligned}\varphi(x) &= \min_{\alpha\in[0,1]} \max_y (\alpha f_1(x) + (1-\alpha) f_2(x) - \alpha f_1(y) - (1-\alpha) f_2(y)) \\ &= \min_{\alpha\in[0,1]} \{\alpha(-|x-1|/2 + |x|) + |x-1|/2 + \max_y(-\alpha|y| - (1-\alpha)|y-1|/2)\}.\end{aligned}$$

Since this function is convex and piecewise linear in y, it attains its maximum at the bounds of the interval $[0,1]$. Therefore, the last formula can be rewritten as

$$\begin{aligned}&\min_{\alpha\in[0,1]} \{-\alpha(|x-1|/2 - |x|) + |x-1|/2 + \max(-\alpha, -(1-\alpha)/2)\} \\ &= |x-1|/2 + \min_{\alpha\in[0,1]} \{-\alpha(|x-1|/2 - |x| + 1/4) + |1-3\alpha|/4 - 1/4\}.\end{aligned}$$

Since the function is piecewise linear, its minimum is attained at the points 0, 1, and 1/3. Hence, the last formula can be expressed as

$$\begin{aligned}&|x-1|/2 + \min\{0, -|x-1|/2 + |x|, (1/3)(-|x-1|/2 + |x| - 1)\} \\ &= |x-1|/2 + \min\{0, (1/3)(-|x-1|/2 + |x| - 1)\}.\end{aligned}$$

This equality holds, because the second term is always less than the third term. Finally, the function can be expressed as

$$\varphi(x) = \begin{cases} (x-1)/2, & x \geqslant 1, \\ 0, & 0 \leqslant x \leqslant 1, \\ -2x/3, & -3 \leqslant x \leqslant 0, \\ (1-x)/2, & x \leqslant -3. \end{cases}$$

The function $\varphi(x)$ is obviously not convex.

3. CONCLUSION

Convolution (7) reduces the vector optimization problem (in the sense of constructing a set of Slater-optimal points) to a scalar optimization problem. Therefore, multicriteria optimization is reduced to computing the criteria convolution (7). The examples given in the paper show that the computation is not an easier problem than the problem of vector optimization. Nevertheless, construction of this convolution is of great interest, because it, being the gain function of the active element, is the obvious convolution for the base model of active systems. Therefore, this convolution, if reformulated as a base model of active systems, is helpful in multicriteria optimization. This reformulation may also yield other interpretations in multicriteria optimization, which require independent investigation.

APPENDIX

Proof of Lemma 1.

$$x \in S \iff x \in X^{[x]}(f) \iff \varphi(x) = f(x,x) = 0.$$

Proof of Lemma 2. The first property is self-evident; therefore it suffices to demonstrate the second property.

1. Let $x \in S(f)$. By virtue of the Karlin lemma, for some $\lambda_i \geqslant 0$: $\sum_{i\in I} \lambda_i = 1$, we have

$$x = \arg\min_{y\in Y} \sum_{i\in I} \lambda_i f_i(y).$$

Consequently, for these λ_i, the subextremal expression in formula (8) is

$$\sum_{i\in I}\lambda_i f_i(x) - \min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y) = 0.$$

Therefore, since $\phi \geqslant 0$, for this x we find that $\phi(x) = 0$.

2. Conversely, let $\phi(x) = 0$. Consequently, for some $\lambda_i \geqslant 0$, $\sum_{i\in I}\lambda_i = 1$, the subextremal expression in formula (8) is equal to

$$\sum_{i\in I}\lambda_i f_i(x) - \min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y) = 0.$$

Hence,

$$x = \arg\min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y).$$

Now, by the Karlin lemma, we find that $x \in S(f)$. This completes the proof of the lemma.

Proof of Theorem 1. First, let us rewrite the function $\phi(x)$ as

$$\phi(x) = \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\left(\sum_{i\in I}\lambda_i f_i(x) - \min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y)\right) = \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\max_{y\in Y}\sum_{i\in I}(\lambda_i(f_i(x) - f_i(y))).$$

Using the minimax theorem [10], which holds because the function $\sum_{i\in I}\lambda_i(f_i(x) - f_i(y))$ is concave in y and convex in λ, we find that

$$\begin{aligned}\phi(x) &= \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\max_{y\in Y}\sum_{i\in I}(\lambda_i(f_i(x) - f_i(y))) = \max_{y\in Y}\min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\sum_{i\in I}(\lambda_i(f_i(x) - f_i(y)))\\ &= \max_{y\in Y}\left\{\min_{i\in I}(f_i(x) - f_i(y))\right\} = \varphi(x).\end{aligned}$$

This completes the proof.

Proof of Corollary 1. Since the functions $f_i(y)$ are convex, the conditions of Theorem 1 are satisfied. Therefore

$$\begin{aligned}\varphi(x) &= \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\left(\sum_{i\in I}\lambda_i f_i(x) - \min_{y\in Y}\sum_{i\in I}\lambda_i f_i(y)\right)\\ &= \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\left(\sum_{i\in I}\lambda_i\langle x - y_i, x - y_i\rangle - \min_{y\in Y}\sum_{i\in I}\lambda_i\langle y - y_i, y - y_i\rangle\right)\\ &= \min_{\lambda_i\geqslant 0,\sum\lambda_i=1}\left\langle x - \sum_{i\in I}\lambda_i y_i, x - \sum_{i\in I}\lambda_i y_i\right\rangle\\ &= \mathrm{dist}^2(x, \mathrm{conv}\{y_1, y_2, \ldots, y_n\}).\end{aligned}$$

Here the symbol $\langle\cdot,\cdot\rangle$ denotes scalar multiplication. This completes the proof.

REFERENCES

1. S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vols. 1, 2, London (1959).
2. Yu. B. Germeier, *Introduction to Operations Research* [in Russian], Nauka, Moscow (1971).
3. V. V. Podinovskii and V. D. Nogin, *Pareto-Optimal Solutions of Multicriteria Problems* [in Russian], Nauka, Moscow (1982).

4. M. Z. Arslanov, "Relationship between consistent planning and multicriteria optimization," in: *Models and Methods of Automatic Production Control Systems* [in Russian], Institute of Engineering and Technology, Almata, Kazakstan (1995), pp. 4–9.
5. M. B. Aidarkhanov, M. Z. Arslanov, and A. B. Murzametov, "Consistent optimization in active systems: Multicriteria nature, fuzziness, and relationship with vector optimization," in: *Problems of Informatics and Control* [in Russian], Gylym, Almata (1995), pp. 3–12.
6. A. A. Ashimov, V. N. Burkov, B. A. Dzhaparov, and V. V. Kondrat'ev, *Consistent Control of Active Production Systems* [in Russian], Nauka, Moscow (1986).
7. V. N. Burkov and V. V. Kondrat'ev, *Functioning Mechanisms of Organizational Systems* [in Russian], Nauka, Moscow (1981).
8. V. N. Burkov, B. Danev, A. K. Enaleev, et al., *Large Systems: Modeling of Organizational Mechanisms* [in Russian], Nauka, Moscow (1989).
9. A. A. Ashimov, M. Z. Arslanov, V. N. Burkov, and B. A. Dzhaparov, "Properties and construction of sets of consistent plans," in: *Uncertainty, Risk, and Dynamics in Organizational Systems* [in Russian], Institute of Control Sciences, Moscow (1984), pp. 45–60.
10. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Elsevier (1979).