REPRESENTATION OF MATRIX CONVOLUTIONS BY GENERALIZED ADDITIVE CONVOLUTIONS IN INTEGRATED EVALUATION

E. V. Umrikhina

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We consider the representation of a matrix (logical) convolution of local evaluations in the form of a generalized additive convolution in integrated evaluation problems. Necessary and sufficient conditions for exact representation are given. Two approaches are proposed to solving the problem of exact representation. An approximate solution is sketched.

1. Introduction

In integrated evaluation problems, the choice of the procedure for aggregation (convolution) of local evaluations into an integrated evaluation constitutes an important independent problem alongside the problems of choosing a system of evaluation measures and developing rules for the application of integrated evaluation in management practice.

The choice of the aggregation technique is determined by the local structure of the system of measures, the interrelationships between local and integrated evaluations, the scales on which the evaluations are expressed, etc.

The suitability of specific aggregation techniques in particular integrated evaluation problems and questions of interchangeability of the different techniques are studied in the theory of active systems. One of the problems, in particular, is the representation of matrix (logical) convolutions by additive convolutions. Additive convolutions are simpler and are widely used in practice, whereas matrix convolutions are easy to visualize, simple to correct, provide a clearer picture of the politics of the person (organ) performing the evaluation, and constitute a universal form of representation of convolutions of any kind [1-3].

The problem of approximate representation of the matrix convolution by an additive convolution with minimum approximation error was presented and solved for a number of particular matrix convolutions in [4]. The maximum approximation error of a monotone matrix convolution by an additive convolution was also estimated in [4], where it was found to be quite substantial.

A fairly simple convolution was proposed in [4], leading to a smaller approximation error than the additive convolution: this is the generalized additive convolution (GAC), i.e., an additive convolution with a nonlinear scale transformation. It was shown in [5] that some matrix convolutions have an exact representation in the form of GAC, while their approximation by an additive convolution gives a substantial error.

Some problems associated with the analysis of GAC were formulated in [4, 5]:

1) exact representability of an arbitrary matrix convolution in GAC form and methods of construction of such GAC;

2) approximate representation of an arbitrary matrix convolution in GAC form with minimum approximation error;

3) estimating the maximum approximation error of an arbitrary matrix convolution in GAC form.

The first two problems are considered in the present paper.

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2. The Problem

Let $C = \|c_{ij}\|$ be a positive monotone logical convolution matrix, $i = \overline{1, n}$, $j = \overline{1, m}$. Each element c_{ij} of this matrix is an evaluation by some generalized measure given that the evaluation by one local measure is k_i^1 and by another k_j^2, $i = \overline{1, n}$, $j = \overline{1, m}$.

Monotonicity of the matrix implies that $c_{ij} \gg c_{ij-1}$ for $i = \overline{1, n}$, $j = \overline{2, m}$ and $c_{ij} \geq c_{i-1j}$ for $i = \overline{2, n}$, $j = \overline{1, m}$.

Monotonicity of the matrix follows from the substantive interpretation of the integrated evaluation problem. As a rule, the evaluation measures are chosen so that a higher value of the measure corresponds to a higher (not lower) evaluation, and a higher evaluation by local measures corresponds to a higher (not lower) integrated evaluation.

We moreover assume that the matrix C has no equal rows and columns, i.e., $\forall_i\ \exists_j\ c_{ij} > c_{i-1j}$, $i = \overline{2, n}$, $j = \overline{1, m}$, for $\forall_j\ \exists_i\ c_{ij} > c_{ij-1}$, $i = \overline{1, n}$, $j = \overline{2, m}$. This property of the matrix is again quite natural: if there are equal rows (columns) in the matrix, the corresponding local evaluations may be combined, reducing the number of scale gradations and passing from the original matrix to a smaller matrix.

In case of generalized additive convolutions, the integrated evaluation by two local measures is $\varphi(u_i + v_j)$, where u_i is the value assigned to the i-th evaluation by the first measure, v_j the value assigned to the j-th evaluation by the second measure, $i = \overline{1, n}$, $j = \overline{1, m}$, u_i, v_j are real numbers, and φ is some nonlinear scale transformation.

The problem of representing a matrix convolution in GAC form involves finding the numbers u_i, $i = \overline{1, n}$, v_j, $j = \overline{1, m}$ and a nondecreasing function $\varphi(w)$, $w = u_i + v_j$, such that

$$\varepsilon = \max_{i,j} |c_{ij} - \varphi(u_i + v_j)| \to \min \tag{1}$$

or

$$\delta = \max_{i,j} \left| 1 - \frac{\varphi(u_i + v_j)}{c_{ij}} \right| \to \min. \tag{2}$$

This formulation combines the first two problems of [4, 5]. Indeed, if ε or δ are 0, we obtain an exact approximation of the matrix convolution by a generalized additive convolution: if $\varepsilon, \delta > 0$, then we obtain a minimum-error approximation.

Let L be the set of index pairs of the matrix elements of C such that $[(i, j), (k, s)] \in L$, if and only if

$$c_{ij} > c_{ks}. \tag{3}$$

Consider the system of linear inequalities

$$u_i + v_j > u_k + v_s \quad \forall [(i, j), (k, s)] \in L. \tag{4}$$

THEOREM 1. For exact representation of a positive monotone matrix $C = \|C_{ij}\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ by a nondecreasing function $\varphi(u_i + v_j)$, u_i, v_j are real numbers, $i = \overline{1, n}$, $j = \overline{1, m}$, it is necessary and sufficient that the system (4) has a solution.

The proof is given in the Appendix.

Thus, by Theorem 1, the original problem can be reduced to solving a system of linear inequalities (4), and if the solution $(\mathbf{u}^*, \mathbf{v}^*)$ exists, then we can always construct a nondecreasing $\varphi(w)$ such that $\varphi(u_i^* + v_j^*) = c_{ij}$.

To this end, arrange the elements c_{ij} in nondecreasing order and construct the corresponding sequence $\{w_{ij}^*\}$, $w_{ij}^* = u_i^* + v_j^*$. Isolate subsequences $\{w_{ij}^*\}_q$ such that each subsequence $\{w_{ij}^*\}_q$ corresponds to the same value $c_{ij}^q = c^q$, $q = \overline{1, Q}$, where Q is the number of values c_{ij}, $c^{q+1} > c^q$. Let Ω_q be the set of index pairs (i, j) corresponding to c^q and $\{w_{ij}^*\}_q$. The subsequences constructed in this way are disjoint because if $c^{q+1} > c^q$, then $(w_{ij}^*)_{q+1} > (w_{ij}^*)_q$ for all $(i, j)_q \in \Omega_q$, $(i, j)_{q+1} \in \Omega_{q+1}$, since $w_{ij}^* = u_i^* + v_j^*$, and $(\mathbf{u}^*, \mathbf{v}^*)$ is a solution of (4). Thus, $\min_{(i,j)\in\Omega_q} w_{ij}^* > \max_{(i,j)\in\Omega_q} w_{ij}^*$.

Therefore the function $\varphi(u_i^*+v_j^*)$ defined by

$$\varphi(u_i^*+v_j^*)=c^q, \text{ if } \min_{(i,j)\in\Omega_q} w_{ij}^*\leqslant w_{ij}^*\leqslant \max_{(i,j)\in\Omega_q} w_{ij}^*, \quad q=\overline{1,Q},$$

is nondecreasing and satisfies the equality $\varphi(u_i^*+v_j^*)=c_{ij}$.

Note a number of properties of solutions of this problem.

Property 1. The sequences $\{u_i\}$, $\{v_j\}$ are strictly increasing, i.e., $u_i > u_{i-1}$, $v_j > v_{j-1}$, $i = \overline{2, n}$, $j = \overline{2, m}$.

Property 2. Let $(\mathbf{u}^*, \mathbf{v}^*)$ be a solution of the system (4). Then $\omega=(a\mathbf{u}^*+\mathbf{b}', a\mathbf{v}^*+\mathbf{b}'')$ is also a solution of the system (4) ($a > 0$, $b_i' = b'_{i+1}$, $i = \overline{1, n} - 1$, $b_j'' = b''_{j+1}$, $j = \overline{1, m} - 1$; b_i', b_j'' are real numbers).

By this property, if there exists a solution $(\mathbf{u}^*, \mathbf{v}^*)$ of the system (4), then there also exists a solution $(\mathbf{u}, \mathbf{v})$, such that $u_1 = v_1 = 0$, u_i, $v_j > 0$, $i = 2, n$, $j = \overline{2, m}$.

Property 3. Let $\mathbf{z}^1=(\mathbf{u}^1, \mathbf{v}^1)$ and $\mathbf{z}^2=(\mathbf{u}^2, \mathbf{v}^2)$ be two solutions of the system (4). Then $\mathbf{z}^3=\alpha\mathbf{z}^1+\beta\mathbf{z}^2$ is also a solution of the system (4) (α, $\beta > 0$).

Property 4. Let C be a symemtric matrix, i.e., $m = n$ and $c_{ij} = c_{ji}$. If there exists a solution of the system (4), then there also exists a symmetrical solution of the system (4), i.e., $u_i^* = v_j^*$, $i = \overline{1, n}$.

Properties 1 and 4 are proved in the Appendix, whereas properties 2 and 3 are checked by substituting ω and $\mathbf{z}^3$ in (4).

Using Property 1, we can somewhat simplify the system of inequalities (4). We introduce the additional constraints

$$u_i>u_{i-1}, \quad i=\overline{2, n},$$
$$v_j>v_{j-1}, \quad j=\overline{2, m}. \tag{5}$$

Consider the set $L' \subset L$ of index pairs $[(i, j), (k, s)] \in L$ such that $i > k$, $j < s$ or $i < k$, $j > s$. The system of inequalities

$$u_i+v_j>u_k+v_s \quad \forall[(i,j),(k,s)]\in L' \tag{6}$$

combined with the inequalities (5) is equivalent to the system (4), yet it is simpler to construct when solving the problems for a particular matrix.

3. Methods of Solving the Exact Representation Problem

The simplest way is to reduce the problem (5), (6) to a basic linear programming problem. To this end, we introduce the optimality criterion

$$u_n+v_m\to\min \tag{7}$$

and rewrite (5), (6) in the form of nonstrict inequalities

$$\begin{aligned}&u_i-u_{i-1}\geqslant\varepsilon, \quad i=\overline{2, n},\\ &v_j-v_{j-1}\geqslant\varepsilon, \quad j=\overline{2, m},\\ &(u_i+v_j)-(u_k+v_s)\geqslant\varepsilon \quad \forall[(i,j),(k,s)]\in L',\\ &u_i, v_j\geqslant 0, \quad i=\overline{1, n}, \ j=\overline{1, m}, \quad \varepsilon>0.\end{aligned} \tag{8}$$

By property 2, the system of inequalities (8) may be replaced by the following system:

$$u_i+v_j-(u_k+v_s)\geqslant 1 \quad \forall[(i,j),(k,s)]\in L', \tag{9}$$

$$\begin{aligned}&u_i\geqslant u_{i-1}+1, \quad v_j\geqslant v_{j-1}, \quad i=\overline{2, n}, j=\overline{2, m},\\ &u_i, v_j\geqslant 0, \quad i=\overline{1, n}, \quad j=\overline{1, m}.\end{aligned} \tag{10}$$

The matrix C is conveniently visualized by a $m \times n$-vertex graph G(C) whose vertices correspond to the matrix elements and arcs to the inequalities (9), directed from larger to smaller element. The constraint system (9)-(10) may include redundant inequalities. These inequalities are removed in the following way: if several arcs leave some vertex of the graph G(C) leading to vertices that correspond to elements of the same row (the same

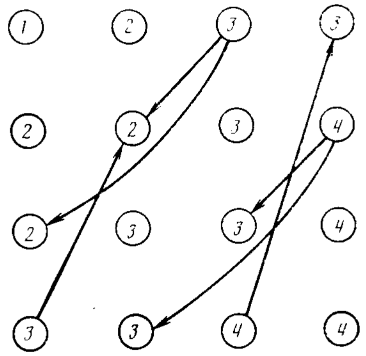

Fig. 1

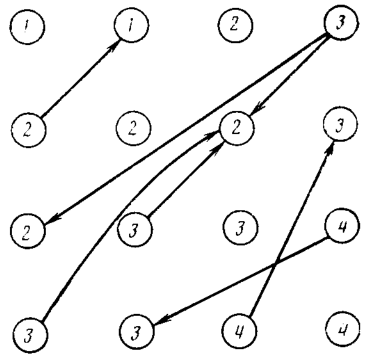

Fig. 2

column), only the arc to the vertex corresponding to the element with the largest column (row) index is retained and the other arcs are removed. If arcs from several vertices corresponding to the elements of the same row (column) enter one node, then only the arc originating from the vertex with the least column (row) index is retained. The inequalities corresponding to the retained arcs, combined with the constraints (10), constitute the minimum constraint system without any redundant inequalities. The inequalities corresponding to the deleted arcs are satisfied automatically given this constainst system.

As an example, consider the matrix

$$C=\begin{pmatrix}1 & 2 & 3 & 3\\ 2 & 2 & 3 & 4\\ 2 & 3 & 3 & 4\\ 3 & 3 & 4 & 4\end{pmatrix}.$$

The graph G'(C) corresponding to the minimum system of inequalities (9) is shown in Fig. 1. The system of inequalities corresponding to the arcs of the graph G'(C) is

$$\begin{gathered}u_1+v_3-(u_2+v_2)\geqslant 1,\\ u_4+v_1-(u_2+v_2)\geqslant 1,\\ u_1+v_3-(u_3+v_1)\geqslant 1,\\ u_4+v_3-(u_1+v_4)\geqslant 1,\\ u_2+v_4-(u_4+v_2)\geqslant 1,\\ u_2+v_4-(u_3+v_3)\geqslant 1.\end{gathered}$$

The solution of (7), (9), (10) in this case is $u_1 = 0$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $v_1 = 0$, $v_2 = 1$, $v_3 = 3$, $v_4 = 5$.

The corresponding function $\varphi(w)$ has the form

$$\varphi(u_i+v_j)=\begin{cases}1, & \text{if} \quad u_i+v_j=0,\\ 2, & \text{if} \quad 1\leqslant u_i+v_j\leqslant 2,\\ 3, & \text{if} \quad 3\leqslant u_i+v_j\leqslant 5,\\ 4, & \text{if} \quad 6\leqslant u_i+v_j\leqslant 8.\end{cases}$$

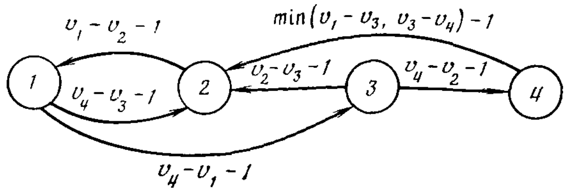

Fig. 3

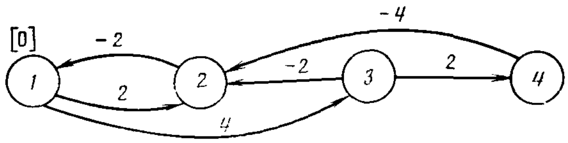

Fig. 4

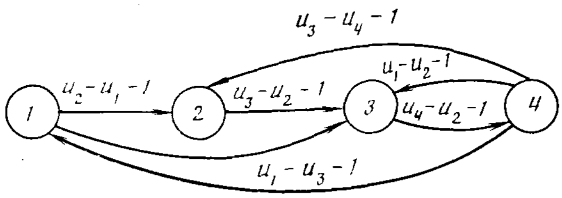

Fig. 5

In general, the problem (7), (9), (10) can be solved by standard linear programming procedures.

Let us consider another approach to the solution of the problem.

Given $\{v_j\}$ it is required to find $\{u_i\}$. From (9) we obtain

$$u_k - u_i \leqslant v_j - v_s - 1 \quad \forall [(i, j), (k, s)] \in L'. \tag{11}$$

Let L_{ik} be the set of pairs (j, s) such that $[(i, j), (k, s)] \in L'$. Define $\ell_{ik} = \min\limits_{(j,s)\in L_{ik}} (v_j - v_s - 1)$. For a n-vertex graph $H(\mathbf{v})$ with arc lengths ℓ_{ik}, $k, i = \overline{1, n}$. Write (11) in the form

$$u_k - u_i \leqslant l_{ik}, \quad k, i = \overline{1, n}. \tag{12}$$

The conditions (12) are typical constraints in the problem of potentials of graph vertices [6]. A necessary and sufficient condition for solubility of the problem of potentials is that the graph has no circuits of negative length.

<u>Example.</u> Let the matrix C have the form

$$C = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}.$$

The graph G'(C) corresponding to the minimum system of inequalities (9) is shown in Fig. 2. The graph $H(\mathbf{v})$ is shown in Fig. 3.

The lengths ℓ_{ik} are shown for the corresponding arcs. The graph $H(\mathbf{v})$ contains three circuits,

$$(1, 2, 1); \ (1, 3, 4, 2, 1); \quad (1, 3, 2, 1).$$

The condition of nonnegativity for the circuits of the graph $H(\mathbf{v})$ has the form

$$\begin{gathered} v_1 - v_2 - v_3 + v_4 \geqslant 2, \\ v_1 - 2v_2 - v_3 + 2v_4 \geqslant 4, \\ -2v_2 + v_3 + v_4 \geqslant 4, \\ -v_4 + v_4 \geqslant 3. \end{gathered}$$

Setting $v_1 = 0$ and using (10), we obtain the solution

$$v_1=0, \quad v_2=1, \quad v_3=2, \quad v_4=5.$$

Let us now determine the potentials of the vertices of the graph $H(\mathbf{v})$ with the arc lengths from Fig. 4. We obtain

$$u_1=0, \quad u_2=2, \quad u_3=4, \quad u_4=6.$$

Thus, the solution of the problem in this case reduces to solving a system of linear inequalities in the variables $\{v_j\}$.

We can similarly construct the graph $H(\mathbf{v})$ which is the dual of $H'(\mathbf{u})$ in a certain sense, and solve the system of inequalities in the variables $\{u_i\}$.

Thus, the graph $H'(\mathbf{u})$ for our example is shown in Fig. 5. This graph has 3 circuits (1, 2, 3, 4, 1), (2, 3, 4, 2), (3, 4, 3).

The corresponding system of inequalities for the circuit lengths has the solution $u_1 = 0$, $u_2 = 2$, $u_3 = 4$, $u_4 = 6$ and the vertex potentials are $v_1 = 0$, $v_2 = 1$, $v_3 = 2$, $v_4 = 5$.

The corresponding function $\varphi(w)$ has the form

$$\varphi(u_i+v_i)=\begin{cases}1, & \text{if } 0\leqslant u_i+v_j\leqslant 1,\\ 2, & \text{if } 2\leqslant u_i+v_j\leqslant 4,\\ 3, & \text{if } 5\leqslant u_i+v_j\leqslant 7,\\ 4, & \text{if } 8\leqslant u_i+v_j\leqslant 11.\end{cases}$$

4. Solution of the Optimal Approximation Problem

If the system (5), (6) is unsolvable for the matrix C, this means that the given matrix cannot be represented exactly by a generalized additive convolution. In this case, we solve the problem of approximate representation with minimum approximation error. Theorem 2 below permits solving this problem by solving a sequence of exact representation problems.

Let L_r'' be the set of index pairs $[(i, j), (k, s)]\in L'$, such that $c_{ij} - c_{ks} > \Delta_r$ and the system

$$u_i>u_{i-1}, \quad v_j>v_{j-1}, \quad i=\overline{2, n}, \quad j=\overline{2, m}, \tag{13}$$

$$u_i+v_j>u_k+v_s \quad \forall [(i, j), (k, s)]\in L_r'' \tag{14}$$

contains no redundant inequalities.

Here $\Delta_r=\min\limits_{L''_{r-1}} \Delta_{ij}^{ks}$, $1\leqslant r\leqslant R$, $\Delta_{ij}^{ks}=c_{ij}-c_{ks}$, $\Delta_0=0$, $\Delta_R\leqslant\max\limits_{L'} \Delta_{ij}^{ks}$.

We will show that $\{\Delta_r\}$ is an increasing sequence. Since $\Delta_r = \min(c_{ij} - c_{ks})$, and for $\forall [(i, j), (k, s)]\in L''$ we have $c_{ij} - c_{ks} > \Delta_{r-1}$, then $\min\limits_{L''_{r-1}}(c_{ij}-c_{ks})>\Delta_{r-1}$, i.e., $\Delta_r > \Delta_{r-1}$, $1 \leq r \leq R$.

THEOREM 2. Let $C = \|c_{ij}\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ be the convolution matrix, r^* the minimal r such that the system (13)-(14) has a solution $(\mathbf{u}^*, \mathbf{v}^*)$, $r^* \neq 0$. Then there exists a nondecreasing function $\varphi^*(w)$ such that $\varepsilon^*=\max\limits_{i,j}|c_{ij}-\varphi^*(u_i^*+v_j^*)|=\min\limits_{(u,v)\varphi}\max\limits_{i,j}|c_{ij}-\varphi(u_i+v_j)|=\Delta_{r^*}/2$.

Theorem 2 is proved in the Appendix.

By Theorem 2, the solution of the problem of approximate representation of the matrix convolution in GAC form with minimum absolute approximation error reduces to sequential solution of the problems (13)-(14) for $0 \leq r \leq r^*$ until for some r^* $(0 < r^* \leq R)$ we obtain the solution $(\mathbf{u}^*, \mathbf{v}^*)$. Defining $\varphi^*(w)$ as in the proof of Theorem 2, we obtain an approximation with minimum absolute error $\varepsilon^* = \Delta_{r^*}/2$.

The algorithm converges in finitely many steps, since in the worst case $(r^*=R,\ \Delta_R=\max\limits_{L'}\Delta_{ij}^{ks})$ the set $L_R''=\varnothing$ and the problem (13)-(14) reduces to the problem (13), which is always solvable.

As an example, consider the matrix

$$C=\begin{pmatrix}1 & 2 & 6\\ 3 & 5 & 7\\ 4 & 8 & 9\end{pmatrix}.$$

TABLE 1

$w_{ij}=w_t$	0	1	2	3	3	4	5	5	7
(i, j)	(1; 1)	(2; 1)	(1; 2)	(2; 2)	(3; 1)	(1; 3)	(2; 3)	(3; 2)	(3; 3)
t	1	2	3	4	5	6	7	8	9
c_t	1	3	2	5	4	6	7	8	9
$\varphi(w_t)$	1	2.5	2.5	4.5	4.5	6	7.5	7.5	9

The system of inequalities (5)-(6) reflecting the conditions of exact representability of this matrix in GAC form is

$$u_3>u_2>u_1, \quad v_3>v_2>v_1,$$

$$u_2+v_1>u_1+v_2, \tag{15}$$

$$u_2+v_2>u_3+v_1, \tag{16}$$

$$u_3+v_2>u_2+v_3, \tag{17}$$

$$u_1+v_3>u_2+v_2. \tag{18}$$

Adding (15) and (16), (17) and (18) and setting $u_1 = v_1 = 0$, we obtain a contradiction.

$$2u_2>u_3, \quad 2u_2<u_3,$$

i.e., C has no exact representation in GAC form, and we therefore seek an optimal approximation with minimum absolute error.

Since $\Delta_1=\min\limits_{L''_0}(c_{ij}-c_{ks})=1$, then L''_1 is the set of pairs [(i, j), (k, s)] such that $c_{ij}-c_{ks} \xrightarrow{L''_0} \Delta_1=1$ and the system (13)-(14) contains no redundant inequalities. The system (13)-(14) of this problem for r = 1, $\Delta_r = 1$ has the form

$$u_3>u_2>u_1, \quad v_3>v_2>v_1,$$

$$u_3+v_1>u_1+v_2,$$

$$u_1+v_3>u_3+v_2,$$

$$u_3+v_2>u_1+u_3.$$

This system has the solution $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $v_1 = 0$, $v_2 = 2$, $v_3 = 4$. The solution is obtained already in step 1, i.e., $r^* = 1$, $\varepsilon^* = \Delta_{r^*}/2 = \Delta_1/2 = 0.5$. The corresponding function $\varphi^*(w)$ is shown in Table 1.

The original matrix C is indeed representable in GAC form with error $\varepsilon^* = 0.5$.

In conclusion note that the results reported in this paper enable us to select a generalized additive convolution which ensures the most accurate representation of the given matrix convolution in specific integrated evaluation problems.

Further theoretical studies of GAC will require estimation of the maximum approximation error of an arbitrary matrix convolution represented in GAC form, analysis of the existence conditions of integral solutions, extension of the results to the case of n local measures.

APPENDIX

Proof of Theorem 1. Necessity. Let $\exists u_i^*, v_j^*, i=\overline{1, n}, j=\overline{1, m}, \varphi^*(w)$, such that $C_{ij}=\varphi^*(u_i^*+v_j^*)$ for $\forall i, j$ and let $[(i, j), (k, s)]\in L$. If $c_{ij} > c_{ks}$, then $\varphi^*(u_i^*+v_j^*)>\varphi^*(u_k^*+v_s^*)$ and by monotonicity of $\varphi^*(w)$, $u_i^*+v_j^*>u_k^*+v_s^*$.

Sufficiency. Let the system (4) have the solution (u^*, v^*). Define the function $\varphi^*(w)$ at the point $w = u_i^* + v_j^*$ as follows: $\varphi^*(u_i^*+v_j^*)=c_{ij}$. $\varphi^*(w)$ is nondecreasing since (u^*, v^*) is a solution of the system (4). Q.E.D.

Proof of Property 1. Since the matrix C has no equal rows, then for $\forall i \ \exists j \ c_{ij}>c_{i-1j}, i=\overline{2, n}$, $j = \overline{1, m}$, and therefore $u_i + v_j > u_{i-1} + v_j$, i.e., $u_i > u_{i-1}$. The proof for columns is similar.

Proof of Property 4. Let $(\mathbf{u}^1, \mathbf{v}^1)$ be a solution of the system (4). Then $\mathbf{u}^2=\mathbf{v}^1$ and $\mathbf{v}^2=\mathbf{u}^1$ is also a solution by symmetry of the matrix C. But then by property 3.

$$\mathbf{u}^* = \frac{\mathbf{u}^1+\mathbf{u}^2}{2} = \frac{\mathbf{u}^1+\mathbf{v}^1}{2}; \quad \mathbf{v}^* = \frac{\mathbf{v}^1+\mathbf{v}^2}{2} = \frac{\mathbf{v}^1+\mathbf{u}^1}{2} = \mathbf{u}^*$$

is also a solution.

Proof of Theorem 2. Let $(\mathbf{u}^*, \mathbf{v}^*)$ be the solution of (13)-(14) obtained for some r^*. Consider the matrix $W^* = \|w^*_{ij}\|$, $w^*_{ij} = u^*_i - v^*_j$. Arrange the elements w^*_{ij} in nondecreasing order and number them consecutively. We obtain the sequence $\{w^*_t\} = (w^*_1, \ldots, w^*_t, \ldots, w^*_T)$, where $w^*_1 = \min_{i,j} w^*_{ij}$, $w^*_T = \max_{i,j} w^*_{ij}$, $T = m \times n$, $w^*_t \geqslant w^*_{t-1}$, $t = \overline{2, T}$. Construct the corresponding sequence of values $\{c_{(ij)_t}\} = \{c_t\}$.

Since $r^* \neq 0$, i.e., the condition of exact representation does not hold for the matrix C, there are pairs (t', t''), $t'' > t'$, such that $c_{t'} > c_{t''}$, but $w^*_{t'} \leq w^*_{t''}$ or $c_{t''} > c_{t'}$, but $w^*_{t''} = w^*_{t'}$. Consider one of these intervals $[t', t'']$.

The interval $[t', t'']$ may be such that $t'' = t' + 1$, i.e., t' and t'' are two neighboring points. But possibly there exist $t \in [t', t'']$, $t \neq t'$, $t \neq t''$. Let τ be the set of pairs $(t^1, t^2) \in [t', t'']$, such that $c_{t^1} > c_{t^2}$, but $w^*_{t^1} \leq w^*_{t^2}$ or $c_{t^2} > c_{t^1}$ but $w^*_{t^2} = w^*_{t^1}$. Let $\Delta_{r_0} = \max_{\tau} |c_{t^2} - c_{t^1}| = |c_{t_0^2} - c_{t_0^1}|$. Since r^* is the minimal r for which (13)-(14) has a solution, and $\{\Delta_r\}$ is an increasing sequence, we have $0 < \Delta_{r_0} \leq \Delta_{r^*}$.

For all $w^*_t, t \in [t', t'']$ let $\varphi^*(w_t^*) = \min(c_{t_0^2}, c_{t_0^1}) + \frac{\Delta_{r_0}}{2}$. In this case, $\varepsilon^*[t'', t'] = \max_{t \in [t'', t']} |c_t - \varphi^*(w^*_t)| = \frac{\Delta_{r_0}}{2} = \min_{\varphi(w_t^*)} \max_{t \in [t'', t']} |c_t - \varphi(w^*_t)|$, since any other nondecreasing $\varphi(w^*_t)$, defined on $[t'', t']$ cannot ensure $\varepsilon[t'', t'] < \varepsilon^*[t'', t']$.

Let t^U be the nearest number greater than t'' such that $c_{t^U} > \min(c_{t_0^2}, c_{t_0^1}) + \frac{\Delta_{r_0}}{2}$, t^L the nearest number smaller than t' such that $c_{t^L} < \min(c_{t_0^2}, c_{t_0^1}) + \frac{\Delta_{r_0}}{2}$. For all $t'' < t < t^U$, $t^L < t < t'$, also define $\varphi^*(w_t^*) = \min(c_{t_0^2}, c_{t_0^1}) + \frac{\Delta_{r_0}}{2}$, then $\varepsilon^*[t^L, t^U] = \max_{t \in [t^L, t^U]} |c_t - \varphi^*(w_t^*)| = \frac{\Delta_{r_0}}{2} = \min_{\varphi(w_t^*)} \max_{t \in [t^L, t^U]} |c_t - \varphi(w_t^*)|$.

$\varphi^*(w_t^*)$ is similarly defined on all intervals of this type.

On all other intervals, where $c_{t''} > c_{t'}$, $w_{t''} > w_{t'}$, or $c_{t''} = c_{t'}$, $w_{t''} \geq w_{t'}$, set $\varphi^* \cdot (w^*_t) = c_t$.

The function $\varphi^*(w^*_t)$ defined in this way is nondecreasing on the entire set of values $\{w^*_t\}$, $1 \leqslant t \leqslant T$, and $\varepsilon^* = \varepsilon^*[1, T] = \max_{t \in [1, T]} |c_t - \varphi^*(w^*_t)| = \frac{\max_{t \in [1, T]} \Delta_{r_0}}{2} = \min_{\varphi(w_t^*)} \max_{t \in [1, T]} |c_t - \varphi(w_t^*)|$.

Since r^* is the minimal r for which (13)-(14) has a solution, then for $r = r^* - 1$ no such solution exists, i.e., there exists at least one index pair $[(i, j), (k, s)]$ such that $[(i, j), (k, s)] \in L''_{r^*-1}$, i.e., $c_{ij} - c_{ks} > \Delta_{r^*-1}$, $c_{ij} - c_{ks} \leq \Delta_{r^*}$ or $c_{ij} - c_{ks} = \Delta_{r^*}$, and $w^*_{ij} \leq w^*_{ks}$.

Let (i, j) correspond to t^1 and (k, s) to t^2, $t^2 > t^1$. Thus, on some interval (t', t'') there exists a pair of points (t^1, t^2), $t^2 > t^1$, such that $c_{t^1} > c_{t^2}$, $c_{t^1} - t_{t^2} = \Delta_{r^*}$.

Since $\{\Delta_r\}$ is an increasing sequence, then $\Delta_{r^*} = \max_{\tau} \Delta_r = \Delta_{r_0}$ for the interval $[t', t'']$ and at the same time $\Delta_{r^*} = \max_{[1, T]} \Delta_r = \max_{[1, T]} \Delta_{r_0}$, so that $\varepsilon^* = \varepsilon^*[1, T] = \min_{\varphi(w_t^*)} \max_{t \in [1, T]} |c_t - \varphi(w^*_t)| = \frac{\Delta_{r^*}}{2}$.

Assume that $(\mathbf{u}^*, \mathbf{v}^*)$ is a nonunique solution of (13)-(14) for $r = r^*$, i.e., there exists another solution $(\mathbf{u}^*, \mathbf{w}^*)$ distinct from $(\mathbf{u}^1, \mathbf{v}^1)$. Construct $\varphi^1(w_t^1)$ similarly to $\varphi^*(w_t^*)$. Then

$$\max_{t\in[1, T]} |c_t - \varphi^1(w_t^1)| = \min_{\varphi(w_t^1)} \max_{t\in[1, T]} |c_t - \varphi(w_t^1)| = \frac{\Delta_{r^*}}{2}, \quad \text{and so} \quad \varepsilon^* = \frac{\Delta_{r^*}}{2} = \min_{(\mathbf{u}, \mathbf{v}), \varphi(w)} \max_{i, j} |c_{ij} - \varphi(u_i + v_j)|. \quad \text{Q.E.D.}$$

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