EVOLVING SYSTEMS

OPTIMAL INCENTIVES IN AN ACTIVE SYSTEM WITH A STOCHASTIC ELEMENT

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A model of an active system with a stochastic element is considered. The optimal incentive function is synthesized.

1. Introduction

The synthesis of an optimal incentive function in active system theory has been studied in detail only for the case of deterministic active systems when the center has complete information about the models of the active elements [1, 2]. For active systems with random disturbances, the optimal incentive function has been synthesized only for the fairly particular case of a linear cost function [3]. An approximate solution of the problem of determining an optimal incentive function for the case of a nonlinear cost function is also given in [3]. The synthesis of an optimal strategy of the center for the case when the choice set of the active element depends on a random parameter is considered in [4].

The synthesis of an optimal incentive function in its general form is a nonstandard extremum problem for some function that depends on the sought incentive function and on the phase states of the system, while the phase states themselves are determined as the solution of an extremal problem which also depends on the sought incentive function.

In this paper, we solve the synthesis problem by reducing it to a classical optimal control problem to which Pontryagin's maximum principle is then applied.

2. Active System Model and Statement of the Problem

Consider a system consisting of a controlling element (the center) and one active element (AE). The AE is a model of some social-economic object whose performance is evaluated by the measure x, $x \geq 0$. For example, x can be identified with a measure of the output or with some integrated performance measure of the AE. The AE objective function has the form

$$f(x, r)=\sigma(x)-\zeta(x, r),$$

where $\sigma(x)$ is the incentive function set by the center and $\zeta(x, r)$ is the cost function of the active element associated with the achievement of the value x of the performance measure. Here r is a random parameter characterizing the production possibilities of the AE. A cost function dependent on the parameter r models the effect of stochastic external disturbances on the object (the active element).

We assume that $r \in [\varepsilon, M]$, where $M > \varepsilon > 0$, and $x \in X(r)$, where X(r) is a set to be defined below.

The objective function of the center will be denoted by $f^0 = f^0(x, \sigma(x))$.

We assume that the active system functions in the following order. First, the center sets the incentive function $\sigma(\cdot)$, given the distribution function F(r) of the random variable r. Then the parameter value r is realized and the AE, given the incentive function and the known realization r of the parameter, selects a value of the performance measure x. The value $x = x(r)$ is chosen by the active element so as to maximize its objective function

$$x(r)\in \operatorname{Arg} \max_{z\in X(r)} f(z,r). \tag{1}$$

Now consider the general problem of synthesizing an optical incentive function $\sigma(x)$. This problem is formulated as follows: For some given class of functions G, determine a function

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$\sigma = \sigma(x)$ such that the expected value of the objective function of the center is maximized subject to the choice by the active element of the value $x = x(r)$ of the performance measure from (1), i.e.,

$$J(\sigma) = \max_{\sigma^0 \in G} \int_{\varepsilon}^{M} \inf_{x(r) \in \tilde{X}(r)} f^0(x(r), \sigma^0(x(r)))\, dF(r), \tag{2}$$

where

$$x(r) \in \tilde{X}(r) = \operatorname{Arg} \max_{z \in X(r)} (\sigma^0(z) - \zeta(z, r)). \tag{3}$$

By specifying the objective function f^0, the cost function $\zeta(x, r)$, the distribution function $F(r)$, and the class of incentive functions, we identify a specific synthesis problem. The following assumptions specialize our problem.

1°. The set G of incentive functions $\sigma(x)$ consists of all twice piecewise-differentiable functions defined on $[0, \infty)$ that satisfy the inequalities $0 \le \sigma(x) \le g$, where g is the incentive "fund."

2°. $F(r)$ is an absolutely continuous distribution function, $F(M) = 1$, $F(\varepsilon) = 0$, where $M > \varepsilon > 0$.

3°. The cost function $\zeta(x, r)$ has the domain of definition $\varepsilon \le r \le M$, $x \in X(r) = [0, x^*(r))$, where $x^*(r)$ is a nondecreasing function of r, and in particular we may have $x^*(r) = \infty$. We assume that $\forall r \in [\varepsilon, M]$, $\zeta(x^*(r), M) > g$.

4°. The cost function $\zeta(x, r)$ is twice continuously differentiable with respect to any of the variables $\zeta(0, r) = 0$, $\dot{\zeta}_x(x, r) > 0$, $\dot{\zeta}_r(x, r) < 0$, $\ddot{\zeta}_{xx}(x, r) > 0$, $\ddot{\zeta}_{xr}(x, r) < 0$ for $x > 0$, $r > 0$. Examples of such cost functions are $\zeta = -\ln(1 - x/r)$, $\zeta = 1/2(x/r)^2$, etc.

5°. $f^0(x, \sigma) = \varphi(x) - \alpha\sigma(x)$, where $\varphi(x)$ is a nondecreasing differentiable function, $\varphi(0) = 0$, $\alpha > 0$.

3. Synthesis of the Incentive Function as the Solution of an Optimal Control Problem

We reduce the optimal incentive function synthesis problem (2), (3) to the classical optimal control problem to which Pontryagin's principle is applicable. We start with some auxiliary propositions, which are proved in the Appendix.

Consider the incentive function $\sigma^1(x)$ such that for some value r_1 of the parameter r we have

$$\sigma^1(x_1) - \zeta(x_1, r_1) = \sigma^1(x_2) - \zeta(x_2, r_1) = \max_{x \in X(r)} [\sigma^1(x) - \zeta(x, r_1)] = M^0, \; x_1 < x_2,$$

and also the incentive function $\sigma^2(x)$ such that

$$\sigma^2(x) = \begin{cases} \sigma^1(x), & \text{if} \quad x \notin [x_1, x_2], \\ \zeta(x, r_1) + M^0, & \text{if} \quad x \in [x_1, x_2]. \end{cases}$$

LEMMA 1. $J(\sigma^1) = J(\sigma^2)$.

Consider the incentive function $\sigma^3(x)$ such that $\exists\, x_1, x_2, \forall x \in [x_1, x_2]$: $\sigma^3(x_1) \ge \sigma^3(x)$, and the function

$$\sigma^4(x) = \begin{cases} \sigma^3(x), & \text{if} \quad x \notin [x_1, x_2], \\ \sigma^3(x_1), & \text{if} \quad x \in [x_1, x_2]. \end{cases}$$

LEMMA 2. $J(\sigma^3) = J(\sigma^4)$.

COROLLARY. The optimal incentive function belongs to the class of nondecreasing functions.

LEMMA 3. For any piecewise-continuous function $\sigma^5(x)$ there is a continuous function $\sigma^6(x)$ such that $J(\sigma^5) \le J(\sigma^6)$.

Lemmas 2 and 3 imply that the optimal incentive function on the class of twice piecewise-differentiable functions belongs to the class of nondecreasing, continuous, twice piecewise-differentiable functions. Therefore, in what follows we assume that $\sigma(x)$ is continuous and nondecreasing.

Let us transform problem (2), (3) to a different form.

A necessary condition of extremum in problem (1) (the choice of the performance measure by the active element) for a continuous piecewise-differentiable function $f(x, r) = \sigma(x) - \zeta(x, r)$ is written in the following form.

If for the given r we have

$$\forall y\colon \sigma(y)<\zeta(y, r), \tag{4}$$

then

$$x(r)=0; \tag{5}$$

otherwise, there exists a choice $x(r) \neq 0$ such that

$$\zeta_x(x, r)\in\partial\sigma(x), \tag{6}$$

where ∂ denotes the subdifferential.

Thus, the solution of problem (2), (4), (5), (6) can be used when looking for a solution of problem (2), (3).

Let us further transform conditions (4), (5), and (6). To this end, we will investigate the solvability for r of the dependence defined by the relationships (4), (5), and (6).

The following lemma is proved in the Appendix.

LEMMA 4. If the cost function ζ has the form $\zeta = \zeta^0(x/r)$, where $\zeta^0(\cdot)$ is a convex, increasing, twice continuously differentiable function defined on the half-open interval $[0, a)$ (here a is a positive number, possibly $a = \infty$),* then

1) the implicit function defined by (4)-(6) is globally solvable for r for all $0 \le x \le x^M$ with the possible exception of finitely many points $x = x_i$, $i = 1, \ldots, I$, where the function $\sigma(x)$ is nondifferentiable, and also with the exception of the point $x = x_0 = 0$; here x^M is defined by conditions (4)-(6) for $r = M$;

2) the function $r = \tilde{r}^0(x, \partial\sigma(x))$ that solves the relationships (4)-(6) is nondecreasing in x;

3) the function $x = \tilde{x}(r)$, which is the inverse of the function $\tilde{r}^0(x, \partial\sigma(x))$, is nondecreasing in r and single valued, with the possible exception of countably many points.

Assume that at the points x, where $\sigma(x)$ is differentiable, we have $u(x) = \dot{\sigma}(x)$, and at the points x_i, $i = 0, 1, \ldots, I$, the function $u(x_i)$ is equal to the left-derivative of the function $\sigma(x)$.

Let

$$\tilde{r}(x, u)=\begin{cases} \tilde{r}^0(x, u), & \text{if} \quad 0\leqslant x\leqslant x^M, \\ M, & \text{if} \quad x>x^M. \end{cases}$$

Suppose that Lemma 4 is true. Then we have the following theorem.

THEOREM 1. The solution of problem (2), (3), for $f^0(x, \sigma) = \varphi(x) - \alpha\sigma(x)$, $G = \{\sigma(x) \mid 0 \le \sigma(x) \le g,\ 0 \le x < \infty\}$ is also the solution of the optimal control problem

$$J(\sigma)=\min_{u=u(x)}\int_0^\infty [F(\tilde{r}(x, u))-1][\dot{\varphi}(x)-\alpha u]dx, \tag{7}$$

*The cost function is taken in the form $\zeta = \zeta^0(x/r)$ because this simplifies the proof of Lemma 4. The same result apparently also holds under weaker conditions on the cost function $\zeta = \zeta(x, r)$.

$$0 \leqslant \int_0^\infty u(x)\,dx \leqslant g, \tag{8}$$

where $u(x) = \dot{\sigma}(x)$.

The proof of Theorem 1 is given in the Appendix.

Problem (7), (8) can be solved by Pontryagin's maximum principle. The Hamiltonian of this problem has the form

$$H(x, u, \lambda) = [1 - F(\tilde{r}(x, u))][\dot{\varphi}(x) - \alpha u] - \lambda u.$$

The optimal control $u(x, \lambda)$ is determined so as to maximize the function H over u, where the parameter $\lambda = \lambda(g) \geq 0$ is calculated from condition (8), i.e., by solving the equation

$$\int_0^\infty u(x, \lambda)\,dx = g.$$

The optimal incentive function is given by

$$\sigma(x) = \int_0^x u(t, \lambda(g))\,dt.$$

In order to compute the optimal incentive function from this formula, we need to know the dependences $u(t, \lambda)$ and $\lambda = \lambda(g)$. Let us investigate the properties of these dependences, which are useful for computing the optimal incentive function.

<u>THEOREM 2</u>. 1) The optimal control $u = u(x, \lambda)$ is a nonincreasing function in the parameter λ. 2) The dependence $\lambda = \lambda(g)$ is decreasing in g.

The proof of Theorem 2 is given in the Appendix.

The monotonicity property of the function $\lambda(g)$ makes it possible to construct iterative algorithms that compute the optimal control by the following scheme.

Take $\lambda > 0$ and find $u(x, \lambda)$ from the condition of maximum of the function $H(x, u, \lambda)$ over u. If $\int_0^\infty u(x, \lambda)\,dx > g$, then in the next iteration λ is incremented by some sufficiently small quantity; if $\int_0^\infty u(x, \lambda)\,dx < g$, then λ is decremented. The procedure is then repeated for the new value of λ until condition (8) is satisfied with sufficient accuracy. The values of λ and $u(x, \lambda)$ obtained in the last iteration are the solution of problem (7), (8).

Let us consider the synthesis of an optimal incentive function in application to a particular example.

Assume that the cost function is $\zeta(x, r) = -\ln(1 - x/r)$, the objective function of the center is $f^0(x, \sigma) = x$, and the random variable r is uniformly distributed on [d, D]. In this case, $\tilde{r}(x, u) = x + 1/u$. From the condition of maximum of the Hamiltonian $H = 1 - F(x + 1/u) - \lambda u$, where F is the uniform distribution function, we obtain

$$\sigma(x) = \begin{cases} -\ln\left(1 - \dfrac{x}{d}\right), & \text{if} \quad 0 \leqslant x \leqslant d - \beta(\lambda), \\ \dfrac{x}{\beta(\lambda)} - \ln\dfrac{\beta(\lambda)}{d}, & \text{if} \quad d - \beta(\lambda) \leqslant x \leqslant D - 2\beta(\lambda), \\ g, & \text{if} \quad D - 2\beta(\lambda) \leqslant x < \infty. \end{cases}$$

Here $\beta(\lambda) = \sqrt{(D - d)\lambda}$.

We see that the solution of the problem consists of three "modes." For certain relationships between the parameters d, D, and g, the first or second mode may be missing.

4. Conclusion

The main result of this paper is the reduction of problem (2), (3) to optimal control problem (7), (8), for which standard solution methods exist. Solution of specific examples of this problem shows that the results may be applied for qualitative and, in some cases, quantitative analysis of incentive systems in the economy.

APPENDIX

Proof of Theorem 1. We will show that if the maximum of the function $f(x, r_1) = \sigma(x) - \zeta(x, r_1)$ for some r_1 is attained at the points x_1 and x_2, $x_1 < x_2$, then for any $r \neq r_1$ the function $f(x, r)$ may not have a maximum at the point $x' \in (x_1, x_2)$.

Assume that this is not so, i.e., there exist x such that

$$\forall x \in X(r): \sigma(x') - \zeta(x', r) \geqslant \sigma(x) - \zeta(x, r). \tag{A.1}$$

The condition of maximum of the function $f(x, r_1)$ at the points x_1 and x_2 has the form

$$\forall x \in X(r): \sigma(x_1) - \zeta(x_1, r_1) = \sigma(x_2) - \zeta(x_2, r_1) \geqslant \sigma(x) - \zeta(x, r_1). \tag{A.2}$$

From (A.1) and (A.2) it follows that

$$\begin{aligned} &\zeta(x_1, r) - \zeta(x', r) \geqslant \zeta(x_1, r_1) - \zeta(x', r_1), \\ &\zeta(x_2, r) - \zeta(x', r) \geqslant \zeta(x_2, r_1) - \zeta(x', r_1). \end{aligned} \tag{A.3}$$

Rewrite (A.3) in the form

$$\int_{x_1}^{x'} \dot{\zeta}_x(x, r)\, dx \leqslant \int_{x_1}^{x'} \dot{\zeta}_x(x, r_1)\, dx, \tag{A.4}$$

$$\int_{x'}^{x_2} \dot{\zeta}_x(x, r)\, dx \geqslant \int_{x'}^{x_2} \dot{\zeta}_x(x, r_1)\, dx. \tag{A.5}$$

From $\ddot{\zeta}_{xr}(x, r) < 0$ it follows that

$$\dot{\zeta}_x(x, r) > \dot{\zeta}_x(x, r_1), \quad \text{if} \quad r_1 > r, \tag{A.6}$$

$$\dot{\zeta}_x(x, r) < \dot{\zeta}_x(x, r_1), \quad \text{if} \quad r_1 < r. \tag{A.7}$$

Inequality (A.6) contradicts inequality (A.4), and inequality (A.7) contradicts inequality (A.5). Our proposition is thus proved.

Note that the functions $\sigma^1(x)$ and $\sigma^2(x)$ coincide over the entire domain of definition with the exception of the closed interval $[x_1, x_2]$. Also note that the maximum of the function $f^2(x, r_1) = \sigma^2(x) - \zeta(x, r_1)$ is attained on the entire closed interval $[x_1, x_2]$. From the proposition that we have just proved it follows that for $r \neq r_1$ the maximum points of the objective functions $f^1(x, r) = \sigma^1(x) - \zeta(x, r)$ and $f^2(x, r) = \sigma^2(x) - \zeta(x, r)$ do not belong to the closed interval $[x_1, x_2]$. But since the functions $\sigma^1(x)$ and $\sigma^2(x)$ coincide outside $[x_1, x_2]$, the maximum points of the functions $f^1(x, r)$ and $f^2(x, r)$ also coincide where $r \neq r_1$. Hence it follows that $J(\sigma^1) = J(\sigma^2)$. Q.E.D.

Proof of Lemma 2. Consider the function $\sigma(x)$ such that $\sigma(x) = \sigma^4(x)$ for $x \notin [x_1, x_2]$ and $\sigma(x) \leq \sigma^4(x)$ for $x \in [x_1, x_2]$, where $x_1 < x_2$. As shown, the maximum point x^* of the function $f(x, r) = \sigma(x) - \zeta(x, r)$ is not an element of (x_1, x_2). Assume that this is not so, $x^* \in (x_1, x_2)$. This leads to the contradictory inequality $\sigma(x^*) - \zeta(x^*, r) \geq \sigma(x_1) - \zeta(x_1, r)$, because $\sigma(x^*) \underline{<} \sigma(x_1)$ by definition of $\sigma(x)$, and $\zeta(x^*, r) > \zeta(x_1, r)$ by monotonicity of the cost function. Thus, $x^* \in (x_1, x_2)$. Hence $J(\sigma) = J(\sigma^4)$. Q.E.D.

Proof of Lemma 3. Let $\sigma^5(x)$ be a nondecreasing function. Otherwise, by Lemma 2, we can always find a nondecreasing function whose effectiveness is not lower.

Let $\sigma^5(x)$ have a discontinuity of the 1st kind at the point $x^* > 0$. The case $x^* = 0$ will be considered below. Denote by $\Delta\sigma^5$ the change of the function $\sigma^5(x)$ at the point x^*.

We will show that there exists a neighborhood $(x^* - \delta, x^*)$ such that the maximum point of the function $f^5(x, r) = \sigma^5(x) - \zeta(x, r)$ does not belong to $(x^* - \delta, x^*)$ for any admissible value $r \in (\varepsilon, M)$. Take δ such that

$$\dot{\zeta}_x(x^*, \varepsilon)\delta<\Delta\sigma^5 \quad (A.8)$$

and $\delta < x^*$, where ε is the least admissible value of the parameter r. Assume that this is not so, i.e.,

$$\exists x'\in(x^*-\delta, x^*), \quad \exists r\in[\varepsilon, M], \quad \forall x: \sigma^5(x')-\zeta(x', r)\geqslant\sigma^5(x)-\zeta(x, r),$$

and so

$$\sigma^5(x')-\zeta(x', r)\geqslant\sigma^5(x^*)-\zeta(x', r). \quad (A.9)$$

By monotonicity of the function $\sigma^5(x)$, we have

$$\sigma^5(x')\leqslant\sigma^5(x^*)-\Delta\sigma^5. \quad (A.10)$$

Summing (A.9) and (A.10), we obtain $\zeta(x^*, r) - \zeta(x', r) \geq \Delta\sigma^5$, i.e., $\int_{x'}^{x^*}\dot{\zeta}_x(x, r)\,dx\geqslant\Delta\sigma^5$.

From $\ddot{\zeta}_{xr}(x, r) < 0$ and $\ddot{\zeta}_{xx}(x, r) > 0$ it follows that $\dot{\zeta}_x(x', r) < \dot{\zeta}_x(x^*, \varepsilon)$. Hence

$$\int_{x'}^{x^*}\dot{\zeta}_x(x_*, \varepsilon)\,dx>\int_{x'}^{x^*}\dot{\zeta}_x(x, r)\,dx\geqslant\Delta\sigma^5$$

or

$$\dot{\zeta}_x(x^*, \varepsilon)\,(x^*-x')>\Delta\sigma^5. \quad (A.11)$$

Clearly, (A.11) contradicts (A.8), because $x^* - x' < \delta$. Thus, there exists a neighborhood $(x^* - \delta, x^*)$ on which the function $f^5(x, r)$ does not attain its maximum for any admissible r.

Consider the function

$$\sigma^6(x)=\begin{cases}\sigma^5(x), & \text{if} \quad x\notin(x^*-\delta, x^*),\\ \dfrac{\sigma^5(x^*+0)-\sigma^5(x^*-\delta)}{\delta}(x-x^*)+\sigma^5(x^*+0), & \text{if} \quad x\in(x^*-\delta, x^*),\end{cases}$$

which is continuous on the closed interval $[x^* - \delta, x^*]$. Note that $\sigma^5(x^* + 0) - \sigma^5(x^* - \delta) \geq \Delta\sigma^5$ by monotonicity of the function $\sigma^5(x)$. Therefore, on the open interval $(x^* - \delta, x^*)$ we have $\dot{\sigma}^6(x) \geq \Delta\sigma^5/\delta > \dot{\zeta}_x(x, \varepsilon) > \dot{\zeta}_x(x, r)$. Hence it follows that the maximum of the function $f^6(x, r) = \sigma^6(x) - \zeta(x, r)$ is not attained on $(x^* - \delta, x^*)$. Since throughout the rest of the domain of definition the functions $\sigma^5(x)$ and $\sigma^6(x)$ coincide, we have $J(\sigma^5) = J(\sigma^6)$. All the discontinuity points of the function $\sigma^5(x)$ are examined similarly.

Now consider the discontinuity point $x^* = 0$, assuming that σ^5 is continuous at all other points. Note that the optimal incentive function at the point $x^* = 0$ takes the value $\sigma(x^*) = 0$. Let $\sigma(0) \neq 0$. Clearly, the active element chooses the same strategy x(r) for any admissible r for the alternative incentive functions $\sigma(x)$ and $\sigma'(x) = \sigma(x) - \sigma(0) \geq 0$. Therefore, in what follows we assume that $\sigma^5(0) = 0$. Since $\zeta(0, \varepsilon)=0$, $\lim_{x\to+0}\sigma^5(x)=\Delta\sigma^5(0)>0$, $\sigma^5(x) \leq g$, $\zeta(x^*(\varepsilon), \varepsilon) > g$, the function $\zeta(x, \varepsilon)$ is continuous, and $\sigma^5(x)$ is continuous for $x > 0$, there exists at least one point x' where $\zeta(x', \varepsilon) = \sigma^5(x')$. Let x' be the least of these points. Define the function $\sigma^6(x)$ as follows: $\sigma^6(x) = \zeta(x, \varepsilon)$ for $0 \leq x \leq x'$ and $\sigma^6(x) = \sigma^5(x)$ for $x > x'$. We will show that $J(\sigma^6) > J(\sigma^5)$. Note that by construction $\sigma^6(x) \leq \sigma^5(x)$. If the x(r) chosen with the incentive function $\sigma^5(x)$ is such that $x(r) \geq x'$, then by the inequality $\sigma^6(x) \leq \sigma^5(x)$ for all $0 \leq x \leq x'$, the same x(r) is chosen for the incentive function $\sigma^6(x)$. If the x(r) chosen for the incentive function $\sigma^5(x)$ is such that $x(r) < x'$, then by property 4° of the cost function the choice with the incentive function $\sigma^6(x)$ is x'. Since $\varphi(x)$ is an increasing function, we have $J(\sigma^6) \geq J(\sigma^5)$. Q.E.D.

Proof of Lemma 4. Consider the solvability for r of the implicit function $\Phi(x, r, \partial\sigma(x)) = \dot{\zeta}_x(x, r) - u(x) - t = 0$, where t is one of the values of the subdifferential $\partial\sigma(x)$. The equation

$$\dot{\zeta}_x(x, r) = u(x) + t \tag{A.12}$$

has at least one solution r^* for some $x^* \neq 0$, t^*. Since $\ddot{\zeta}_{xr}(x, r) < 0$, Eq. (A.12) is locally solvable in the neighborhood of the point (x^*, t^*). If $\zeta(x, r) = \zeta^0(x, r)$, Eq. (A.12) is globally solvable,

$$r = \frac{x}{\mu[x(u(x)+t)]}, \tag{A.13}$$

where $\mu[\cdot]$ is the inverse of the function $\nu(x/r) = \dot{\zeta}_x{}^0(x/r)x$. The function $\nu(x/r)$ is globally invertible because $\dot{\nu}_s(s) = \zeta_{ss}{}^0 s + \dot{\zeta}_s{}^0(s) > 0$ for $s = x/r > 0$. This proves part 1 of the lemma.

We will show that $\tilde{r}(x, \partial\sigma(x))$ is a nondecreasing function, i.e., if $x_1 < x_2$, then $r_1 = \tilde{r}(x_1, t_1(x_1)) < r_2 = \tilde{r}(x_2, t_2(x_2))$, where $t_1(x_1)$ and $t_2(x_2)$ are some values of the subdifferential at the points x_1 and x_2, respectively.

Assume that this is not so, i.e., $x_1 < x_2$ implies that $r_1 > r_2$. From $\sigma(x_1) - \zeta(x_1, r_1) > \sigma(x_2) - \zeta(x_2, r_1)$ and $\sigma(x_2) - \zeta(x_2, r_2) > \sigma(x_1) - \zeta(x_1, r_2)$ we obtain $\zeta(x_2, r_1) - \zeta(x_1, r_1) > \zeta(x_2, r_2) - \zeta(x_1, r_2)$. Hence

$$\int_{x_1}^{x_2} [\dot{\zeta}_x(x, r_1) - \dot{\zeta}_x(x, r_2)]dx > 0. \tag{A.14}$$

But from $\ddot{\zeta}_{xr}(x, r) < 0$ it follows that

$$\dot{\zeta}_x(x, r_1) - \dot{\zeta}_x(x, r_2) < 0 \quad \text{for} \quad r_1 > r_2. \tag{A.15}$$

But (A.15) contradicts (A.14), and so $\tilde{r}(x, \partial\sigma(x))$ is a nondecreasing function. This proves part 2 of the lemma.

Let x_1 and x_2 be such that $r^* = \tilde{r}(x_1, u(x_1)) = \tilde{r}(x_2, u(x_2))$. Then by Lemma 1, there exists an incentive function σ' whose effectiveness is not lower than that of the original function and such that $\tilde{r}(x, u'(x)) = r^*$ for all $x \in [x_1, x_2]$. Define $[\underline{x}_1, \overline{x}_2]$, where $\underline{x}_1 = \inf x$, $\overline{x}_2 = \sup x$ over all x, such that $\tilde{r}(x, u'(x)) = r^*$. As a result we obtain an open interval $(\underline{x}_1, \overline{x}_2)$ where $\tilde{r}$ is constant. Find all such open intervals. There are clearly at most countably many such intervals. Therefore, the inverse of the function $\tilde{r}$ is many-valued only at countably many points. Q.E.D.

Proof of Theorem 1. Consider the function $\tilde{r}(x, \partial\sigma(x))$ and its inverse $\tilde{x}(r)$. The set of definition of the function $\tilde{x}(r)$ is denoted by Ω and its value set by X. By Lemmas 1 and 4, in order to solve the original problem (2), (3) it suffices to consider incentive functions $\sigma(x)$ for which $\tilde{x}(r)$ is such that X is a connected set.

Define the following subsets of the set Ω: Ω_1 the subset of open intervals $(r_j{}^L, r_j{}^U)$ on which the function $\tilde{x}(r)$ is continuous and strictly monotone increasing, $j = 1, \ldots, J_1$; Ω_2 the subset of intervals $(r_k{}^L, r_k{}^U)$ on which the function $\tilde{x}(r)$ is constant, $k = 1, \ldots, K$; $\Omega_3 = \Omega\setminus(\Omega_1 \cup \Omega_2)$. Note that by Lemma 4 Ω_3 consists of at most countably many points r_n, $n = 1, \ldots, N$, and the sets Ω_1 or Ω_2 in general may be empty. We similarly define the value subsets of the function $\tilde{x}(r)$: $X_1 = \tilde{x}(\Omega_1)$; $X_2 = \tilde{x}(\Omega_2)$; $X_3 = \tilde{x}(\Omega_3)$. Note that X_1 is the set of open intervals $(x_j{}^L, x_j{}^U)$, where $x_j{}^L = \tilde{x}(r_j{}^L + 0)$, $x_j{}^U = \tilde{x}(r_j{}^U - 0)$; X_2 is the set consisting of at most countably many points x_k, $k = 1, \ldots, K$; X_3 is the set of closed intervals $[x_n{}^L, x_n{}^U]$, where $x_n{}^L = \tilde{x}(r_n - 0)$, $x_n{}^U = \tilde{x}(r_n + 0)$. Also note that $r_k{}^L = \tilde{r}(x_k - 0, u(x_k - 0))$, $r_k{}^U = \tilde{r}(x_k + 0, u(x_k + 0))$, $r_j{}^L = \tilde{r}(x_j{}^L + 0, u(x_j{}^L + 0))$, $r_j{}^U = \tilde{r}(x_j{}^U - 0, u(x_j{}^U - 0))$, $r_n = \tilde{r}(x_n{}^L + 0, u(x_n{}^L + 0)) = \tilde{r}(x_n{}^U - 0, u(x_n{}^U - 0))$.

Compare two quantities: $I = \int_\Omega [\varphi(x(r)) - \alpha\sigma(x(r))]d[F(r) - 1]$ and $J = \int_X [F(\tilde{r}(x, u(x))) - 1][\dot{\varphi}(x) - \alpha u(x)]dx$.

Note that the corresponding integrals vanish on the subsets Ω_3 and X_2, because Ω_3 and X_2 are sets of measure zero.

Using the properties of the function $\tilde{r}(x, u)$ and $x(r)$, integrating by parts, and changing the variables on the subsets Ω_1 and X_1, we rewrite the expressions for I and J in the form

$$I=\sum_{j=1}^{J_1}[\varphi(x_j^U)-\alpha\sigma(x_j^U)][F(r_j^U)-1]-\sum_{j=1}^{J_1}[\varphi(x_j^L)-\alpha\sigma(x_j^L)][F(r_j^L)-1]+$$
$$+\sum_{k=1}^{K}[\varphi(x_k)-\alpha\sigma(x_k)][F(r_k^U)-1]-\sum_{k=1}^{K}[\varphi(x_k)-\alpha\sigma(x_k)][F(r_k^L)-1]-$$
$$-\sum_{j=1}^{J_1}\int_{x_j^L}^{x_j^U}[F(\tilde r(x,u(x)))-1][\dot\varphi(x)-\alpha u(x)]\,dx,$$
$$J=\sum_{n=1}^{N}[F(r_n)-1][\varphi(x_n^U)-\alpha\sigma(x_n^U)]-\sum_{n=1}^{N}[F(r_n)-1]\times$$
$$\times[\varphi(x_n^L)-\alpha\sigma(x_n^L)]+\sum_{j=1}^{J_1}\int_{x_j^L}^{x_j^U}[F(\tilde r(x,u(x)))-1][\dot\varphi(x)-\alpha u(x)]\,dx.$$

Denote $B(x) = [\varphi(x) - \alpha\sigma(x)][F(\tilde r(x, u(x))) - 1]$ and consider the sum I + J. Reducing and changing the notation, we obtain

$$I+J=\sum_{j=1}^{J_1}[B(x_j^U-0)-B(x_j^L+0)]+\sum_{k=1}^{K}[B(x_k+0)-B(x_k-0)]+$$
$$+\sum_{n=1}^{N}(B(x_n^U-0)-B(x_n^L+0)].$$

From the definition of the sets Ω_1, Ω_2, Ω_3 and respectively the sets X_1, X_2, X_3, it follows that $X = X_1 \cup X_2 \cup X_3$. The points x_j^L, x_j^U, x_k, x_n^L, and x_n^U, where $j = 1, \ldots, J_1$, $k = 1, \ldots, K$, and $n = 1, \ldots, N$, partition the connected set X into a system of at most countably many open and closed intervals. Note that since these points are the boundaries of the corresponding adjacent open or closed intervals, some of these points coincide. Arrange these points in increasing order and denote them by x_i, $i = 1, 2, \ldots$. Represent the sum I + J as $I+J=\sum_i \Delta B(x_i)+B(x^M)-B(x^0)$, where $i=1, 2, \ldots$. It is easy to see that $\Delta B(x_i) = B(x_i - 0) - B(x_i + 0) = 0$, if $\exists\, j, n$ such that $x_i = x_j^L = x_n^U$ or $x_i = u_j^U = x_n^L$, and also $\Delta B(x_i) = B(x_i - 0) + B(x_i + 0) - B(x_i - 0) - B(x_i + 0)$, if $\exists\, j, k, n$ such that $x_i = x_j^U = x_k = x_n^L$ or $x_i = x_n^U = x_k = x_j^L$, or $\exists\, j_1, j_2, k$ such that $x_i = x_{j_1}^U = x_k = x_{j_2}^L$, or $\exists\, n_1, n_2, k$ such that $x_i = x_{n_1}^U = x_k = x_{n_2}^L$; $B(x^0)=\lim_{\delta\to 0} B(x(\delta))$. Note that $B(x^M) = B(x(M)) = [\varphi(x^M) - \alpha\sigma(x^M)][F(M) - 1] = 0$, because $F(M) = 1$. The quantity $B(x^0)$ is also zero. Indeed, as we have shown in the proof of Lemma 3, $\sigma(0) = 0$ and for all $x > 0$, we have $\sigma(x) < \zeta(x, \delta)$ if $\delta < \varepsilon$. Then from (4) and (5) it follows that $x(\delta) = 0$. From $\varphi(0) = 0$ and $\sigma(0) = 0$, we obtain $B(x^0) = B(0) = 0$. Hence it follows that I + J = 0. For $x \geq x^M$ we have $F(\tilde r(x, u)) = 1$, and so $J=\int_0^\infty [F(\tilde r(x,u(x)))-1][\dot\varphi(x)-\alpha u(x)]\,dx.$ Q.E.D.

Proof of Theorem 2. Consider the Hamiltonian $H(x, u, \lambda)$ for two different values of the parameters $\lambda = \lambda' \neq 0$ and $\lambda = \lambda'' \neq 0$, $\lambda' < \lambda''$. By optimality of the controls $u(x, \lambda')$ and $u(x, \lambda'')$ it follows that

$$H(x,u(x,\lambda'),\lambda')\geqslant H(x,u(x,\lambda''),\lambda'),$$
$$H(x,u(x,\lambda''),\lambda'')\geqslant H(x,u(x,\lambda'),\lambda'').$$

Summing these inequalities, we obtain

$$H(x,u(x,\lambda'),\lambda)-H(x,u(x,\lambda'),\lambda'')\geqslant H(x,u(x,\lambda''),\lambda')-$$
$$-H(x,u(x,\lambda''),\lambda'').$$

Hence, substituting the expressions for $H(x, u, \lambda)$, we obtain

$$\lambda''u(x,\lambda')-\lambda'u(x,\lambda')\geqslant\lambda''u(x,\lambda'')-\lambda'u(x,\lambda''),$$

i.e., $u(x, \lambda') \geq u(x, \lambda'')$. This proves part 1 of the theorem.

Let $0 < \lambda' < \lambda''$. Consider the following quantities:

$$g(\lambda') = \int_0^\infty u(x, \lambda')\,dx \quad \text{and} \quad g(\lambda'') = \int_0^\infty u(x, \lambda'')\,dx.$$

From $u(x, \lambda') \geq u(x, \lambda'')$ it follows that $g(\lambda') \geq g(\lambda'')$, i.e., the function $g(\lambda)$ is nonincreasing.

We will now show that $\lambda(g)$ is decreasing in g_∞. Let $g_1 > g_2$ and $\lambda(g_1) = \lambda(g_2) = \lambda^0$, then $g_1 = \int_0^\infty u(x, \lambda^0)\,dx$ and $g_2 = \int_0^\infty u(x, \lambda^0)\,dx$, i.e., $g_1 = g_2$, a contradiction to the assumption $g_1 > g_2$. Thus, the function $\lambda(g)$ is not a constant. Since $g(\lambda)$ is nonincreasing and $\lambda(g)$ is nonconstant, it follows that $\lambda(g)$ is a decreasing function. Q.E.D.

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APPLICATION OF STRATEGIES WITH REFINEMENT IN MULTISTEP CONFLICTS UNDER RISK

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We solve the decision-making problem in a two-level dynamic control system. System dynamics is assumed to depend on random parameters and the elements on different levels have different information about parameter realization. An information exchange procedure is proposed, so that the upper-level element can utilize the information available to the lower-level element. An optimal strategy of the upper-level element is constructed. An example is analyzed.

1. Introduction

Studies of decision-making processes in hierarchical dynamic systems have led to several classes of hierarchical games [1] with different strategies for the two players. Analysis of dynamic conflicts under risk, i.e., in the presence of random parameters in the system, has focused on positional strategies with "memory" [2] and positional counter-strategies with "memory" [3]. It has been noted that under these strategies the player making the first move cannot fully utilize the information available to his partner. It is therefore relevant to consider a wider class of what we call strategies with refinement, similar to the strategies in [4]. Constructs similar to strategies with refinement are also used in the theory of active systems [5].

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