COALITIONS UNDER A COMPETITION MECHANISM OF RESOURCE ALLOCATION

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A competition mechanism of resource allocation to n users was considered in [1]. It was shown that, first, the competition mechanism ensures the existence of a Nash equilibrium in the system, which is determined by the strategy of the "best" loser in the competition, and, second, the competition mechanism achieves a nearly optimal allocation. However, we assumed in [1] that the winners and losers did not form coalitions. In this paper, we investigate the conditions for the formation of such coalitions and their impact on the efficiency of the system.

1. Introduction

An active system with a competition mechanism of resource allocation was described in detail in [1], and here we only review the main notation. We consider a system that consists of a center and n elements. Each element is characterized by its own production function $\varphi_i(x_i)$, which satisfies the following conditions: 1) $\varphi_i(0)=0$, $i=1,\ldots,n$; 2) $\varphi_i(x_i)$ is convex, continuously differentiable, and increasing in x_i, where x_i is the quantity of the resource allocated to element i. We assume that the total quantity of the resource available at the center is R and that the center does not know the exact form of the functions $\varphi_i(\cdot)$.

The system is assumed to function according to the following scheme. The elements forward to the center their requests for the resource s_i and their estimates of resource utilization efficiency ξ_i, where $\xi_i = w_i/s_i$, w_i is the estimated output. The center orders the efficiency estimates ξ_i in a decreasing sequence

$$\xi_{i_1}\geqslant\xi_{i_2}\geqslant\ldots\geqslant\xi_{i_{n-1}}\geqslant\xi_{i_n}, \tag{1}$$

and the first n − 1 winners are declared the winners of the competition. Here $Q = \{i_1, i_2, \ldots, i_{n-1}\}$ is the set of winners (for simplicity, we assume that there is only one loser in the competition).

The resource is then allocated by the following planning procedure:

$$x_i(s_i,\xi_i)=\begin{cases} s_i, & \text{if } i\in Q,\\ c, & \text{if } i\notin Q,\end{cases} \tag{2}$$

where c is selected by some iterative procedure from the balance condition

$$\sum_{i\in Q} s_i+c=R. \tag{3}$$

We assume that the center and the elements attempt to maximize their respective objective functions

$$W_c=\sum_{i=1}^{n}\varphi_i(x_i),\quad W_{i_n}=\varphi_{i_n}(c),$$
$$W_i=\varphi_i(s_i)-\psi_i(\xi_i s_i-\varphi_i(s_i)),\ \psi_i(y)=\begin{cases} 0, & \text{if } y\leqslant 0,\\ \alpha y, & \text{if } y>0.\end{cases}$$

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Let

$$s_i(\xi_i) \in \operatorname{Arg}\max_{s_i} [\varphi_i(s_i) - \psi_i(\xi_i s_i - \varphi_i(s_i))],$$

$$h_i(\xi_i) = \max_{s_i} [\varphi_i(s_i) - \psi_i(\xi_i s_i - \varphi_i(s_i))],$$

$$v_i(c) : h_i(v_i(c)) = \varphi_i(c).$$

Given the properties of the functions $\varphi_i(\cdot)$, we can assert (see lemma in the Appendix) that $s_i(\xi_i)$ decreases in ξ_i for any $i = 1, \ldots, n$.

We have shown [1] that a Nash equilibrium $\xi_i^* = \xi_{i_n}$ exists in the system, i.e., ξ_{i_n} entirely determines the allocation of the resource. It is therefore important to consider the selection of ξ_{i_n} by the loser, and in particular the effect of a coalition between the winners and the loser on the choice of ξ_{i_n}.

2. Coalitions Between the Winners and the Loser and Their Impact on System Operating Efficiency

It is clearly disadvantageous to the loser to report $\xi_{i_n} > v_{i_{n-1}}(c)$. As ξ_{i_n} increases, the loser gets a larger allocation c. The loser will accordingly report $\xi_{i_n} = v_{i_{n-1}}(c)$ and the maximum allocation that the loser may receive is determined from the condition

$$\sum_{i\in Q} s_i(v_{i_{n-1}}(c_{\max})) + c_{\max} = R. \tag{4}$$

Let $\xi_{i_n}^{\max} = v_{i_{n-1}}(c_{\max})$.

Now assume that the winners enter into a coalition with the loser, who is persuaded to lower the estimate ξ_{i_n}. As a result, the winners get an additional allocation. But for the coalition to be beneficial to the loser, the winners should compensate the loser for the decrease of the loser's objective function (due to the decrease of c).

Two compensation techniques are possible.

Case A. The winners pass part of their resource allocation to the loser.

Case B. The winners pass part of the payoff to the loser.

Let us consider each case separately.

Assume that the loser reports some estimate $\tilde{\xi}_{i_n} < \xi_{i_n}^{\max}$ and is allocated a quantity $\tilde{c} < c_{\max}$ (according to condition (3)), i.e.,

$$\sum_{i\in Q} s_i(\tilde{\xi}_{i_n}) + \tilde{c} = R. \tag{5}$$

In case A, the winners should transfer to the loser a quantity $\Delta \geqslant c_{\max} - \tilde{c}$ of the resource (otherwise the coalition is not beneficial to the loser). In other words, we should have

$$\sum_{i\in Q} \Delta_i \geqslant c_{\max} - \tilde{c}, \tag{6}$$

where Δ_i is the quantity of the resource transferred to the loser by winner i.

Using the result of [1] which states that $\{s_i(\xi_{i_n}^{\max})\}$ is the optimal allocation of the resource $R - c_{\max}$ among the winners and condition (6), we can assert that, in general, a coalition between the winners and the loser in case A may lower the operating efficiency of the system. In the particular case when $\sum_{i\in Q} \Delta_i = c_{\max} - \tilde{c}$, $\Delta_i = s_i(\tilde{\xi}_{i_n}) - s_i(\xi_{i_n}^{\max})$ $i \in Q$, the allocations before and after coalition formation are identical, and the efficiency is therefore the same in both cases. However, as we shall see below, no coalition is formed between the winners and the loser in case A, so that system efficiency is not reduced.

We assume that no coalition is formed between the winners and the loser if the payoff of at least one winner after the formation of the coalition is less than the payoff without the coalition.

THEOREM 1. No coalition is formed between the winners and the loser in case A.

The theorem is proved in the Appendix.

Note that we can similarly show that no coalition is formed in case A between the loser and some of the winners. It is thus disadvantageous to the winners to form a coalition transferring part of their allocation to the loser.

However, a different coalition-forming mechanism is possible, in which the winners transfer part of their payoffs to the loser (case B). As before, we assume that the loser reports some estimate $\tilde{\xi}_{i_n} < \xi_{i_n}^{max}$ and is allocated the quantity $\tilde{c}$ according to condition (5). The payoff of the loser without a coalition is

$$\overline{W}_{i_n} = \varphi_{i_n}(c_{\max}),$$

and after coalition formation

$$\tilde{W}_{i_n} = \varphi_{i_n}(\tilde{c}).$$

For the coalition to be beneficial to the loser, the winners must transfer to the loser a payoff $\Delta \geqslant \varphi_{i_n}(c_{\max}) - \varphi_{i_n}(\tilde{c})$, so that the winner payoffs before and after coalition formation are respectively given by

$$\overline{W}_i = \varphi_i(s_i(\xi_{i_n}^{\max})) - \psi_i(\xi_{i_n}^{\max} s_i(\xi_{i_n}^{\max}) - \varphi_i(s_i(\xi_{i_n}^{\max}))),$$

$$\tilde{W}_i = \varphi_i(s_i(\tilde{\xi}_{i_n})) - \psi_i(\tilde{\xi}_{i_n} s_i(\tilde{\xi}_{i_n}) - \varphi_i(s_i(\tilde{\xi}_{i_n}))) - \Delta_i, \; i \in Q$$

where $\sum_{i \in Q} \Delta_i = \Delta$.

As before, we naturally assume that no coalition is formed if for at least one of the winners

$$\tilde{W}_i < \overline{W}_i. \qquad (7)$$

Analysis of more specific examples has shown that in case B a coalition is indeed possible between the winners and the loser. However, we have the following proposition.

THEOREM 2. In Case B, a coalition between the winners and the loser does not reduce the operating efficiency of the system if

1) "strong" penalties are imposed in the system;

2) in case of "weak" penalties,

$$(1+\alpha) s_i \varphi_i''(s_i) + \varphi_i'(s_i) \geqslant 0$$

for all $i \in Q$.

Condition 2 of Theorem 2 may be regarded as a constraint that defines a certain set of values of the parameter α. Let us estimate this set.

Define $\xi_{i_n}^{min}$ such that

$$\sum_{i \in Q} s_i(\xi_{i_n}^{\min}) = R,$$

i.e., the loser gets $c = 0$ if the reported estimate is $\xi_{i_n}^{min}$.

COROLLARY. Condition 2 of Theorem 2 holds for all α satisfying the inequalities

$$0 \leqslant \alpha \leqslant \alpha_U,$$

where

$$\alpha_U = -\max_{i \in Q} \max_{s_i(\xi_{i_n}^{\max}) \leqslant s_i \leqslant s_i(\xi_{i_n}^{\min})} \frac{s_i \varphi_i''(s_i) + \varphi_i'(s_i)}{s_i \varphi_i''(s_i)} > 0.$$

Theorem 2 and the Corollary are proved in the **Appendix**.

A coalition between the loser and some of the winners (even one winner) is also possible in case B. Theorem 2 is true in this case also.

Unfortunately, we could not derive sufficiently general conditions of coalition formation in case B, and we accordingly consider the following example as an illustration.

3. Example

Consider a system consisting of a center and n elements whose production functions have the form $\varphi_i(x_i)=\sqrt{r_i x_i}$, r_i, where r_i are parameters. For definiteness, we assume that $r_1 \geqslant \ldots \geqslant r_{n-1} > r_n$.

Using the definitions of the functions $s_i(\xi_i)$, $h_i(\xi_i)$, and $v_i(c)$, we can easly show that

$$s_i(\xi_i)=\begin{cases}\dfrac{(1+\alpha)^2 r_i}{4\alpha^2\xi_i^2}, & \text{if } \alpha\leqslant 1,\\ \dfrac{r_i}{\xi_i^2}, & \text{if } \alpha\geqslant 1;\end{cases}$$

$$h_i(\xi_i)=\begin{cases}\dfrac{(1+\alpha)^2 r_i}{4\alpha\xi_i}, & \text{if } \alpha\leqslant 1,\\ \dfrac{r_i}{\xi_i} & \text{if } \alpha\geqslant 1;\end{cases}$$

$$v_i(c)=\begin{cases}\dfrac{(1+\alpha)^2}{4\alpha}\sqrt{\dfrac{r_i}{c}}, & \text{if } \alpha\leqslant 1,\\ \sqrt{\dfrac{r_i}{c}}, & \text{if } \alpha\geqslant 1.\end{cases}$$

From the expression for $v_i(c)$ we see that the loser is the element n, because $v_n(c) < v_i(c)$, $i = 1, \ldots, n-1$.

Let us investigate the model for $\alpha\geqslant 1$.

From (4) we obtain

$$c_{\max}=R-H/\xi_{i_n}^{\max 2}=R\Big/\left(1+\frac{H}{r_{n-1}}\right).$$

where

$$H=\sum_{i=1}^{n-1} r_i, \quad \xi_{i_n}^{\max}=\sqrt{\frac{H}{R-c_{\max}}}=\sqrt{\frac{H+r_{n-1}}{R}}.$$

Similarly

$$\tilde{\xi}_{i_n}=\sqrt{\frac{H}{R-\tilde{c}}},$$

and so

$$h_i(\tilde{\xi}_{i_n})=r_i\sqrt{\frac{R-\tilde{c}}{H}}, \quad h_i(\xi_{i_n}^{\max})=r_i\sqrt{\frac{R-c_{\max}}{H}}, \quad i=1,\ldots,n-1.$$

From the expression for $\tilde{W}_i$, $\overline{W}_i$, Δ, and condition (7) it follows that the coalition is beneficial for all elements if

$$\sum_{i=1}^{n-1}[h_i(\tilde{\xi}_{i_n})-h_i(\xi_{i_n}^{\max})]\geqslant\varphi_{i_n}(c_{\max})-\varphi_{i_n}(\tilde{c})$$

or

$$\sqrt{H(R-c_{\max})}\left[\sqrt{\frac{R-\tilde{c}}{R-c_{\max}}}-1\right]\geqslant\sqrt{r_n c_{\max}}-\sqrt{r_n\tilde{c}}. \tag{8}$$

Consider the limiting case when $r_{n-1} \gg r_n$ or $H \gg r_n$. Denote $\delta = c_{max} - \tilde{c}$. Since

$$R-c_{\max}=R\frac{H}{H+r_{n-1}},$$

then using the Taylor formula we obtain

$$\sqrt{H(R-c_{\max})}\left[\sqrt{\frac{R-\tilde{c}}{R-c_{\max}}}-1\right]\simeq\frac{\delta}{2}\sqrt{\frac{H+r_{n-1}}{R}}$$

and so inequality (8) holds for sufficiently large H.

We will show that the converse situation is also possible, i.e., no coalition is formed.

Let $r_{n-1} = r_n$ and $n = 2$, then

$$h_1(\xi_2)-h_1(\xi_2^{\max})=\sqrt{r_1(R-c_{\max})}\left[\sqrt{\frac{R-\tilde{c}}{R-c_{\max}}}-1\right]\leqslant$$

$$\leqslant\sqrt{\frac{r_1}{2R}}\,(c_{\max}-\tilde{c})<\sqrt{\frac{2c_{\max}r_1}{R}}\,(\sqrt{c_{\max}}-\sqrt{\tilde{c}})=$$

$$=\sqrt{r_1}\,(\sqrt{c_{\max}}-\sqrt{\tilde{c}})=\sqrt{r_2}\,(\sqrt{c_{\max}}-\sqrt{\tilde{c}})=\varphi_2(c_{\max})-\varphi_2(\tilde{c}),$$

and no coalition is formed.

In case $\alpha\leqslant 1$, the results are fundamentally the same as in case $\alpha\geqslant 1$.

4. Conclusion

The analysis of this paper shows that under a competition mechanism of resource allocation, a coalition between the winners and the loser is possible only if the winners transfer part of their payoffs to the loser. In some cases, this coalition does not reduce the operating efficiency of the system (Theorem 2). If the winners are required to transfer part of their allocation to the loser, no coalition is formed.

APPENDIX

LEMMA. For any $i = 1, \ldots, n$, the function $s_i(\xi_i)$ is decreasing in ξ_i.

Proof. Define $\tilde{s}_i(\xi_i)$ by the condition $\xi_i\tilde{s}_i(\xi_i) = \varphi_i(\tilde{s}_i(\xi_i))$. Clearly, $s_i(\xi_i)\geqslant\tilde{s}_i(\xi_i)$, i.e., we have the inequality

$$\xi_i s_i(\xi_i)\geqslant\varphi_i(s_i(\xi_i)). \tag{A.1}$$

The proof is by contradiction, i.e., assume that for some ξ_i' and ξ_i'', $\xi_i' < \xi_i''$, we have

$$s_i(\xi_i')\leqslant s_i(\xi_i''). \tag{A.2}$$

From the definition of the function $s_i(\xi_i)$, we have

$$\varphi_i(s_i(\xi_i'))-\psi_i(\xi_i's_i(\xi_i')-\varphi_i(s_i(\xi_i')))>$$
$$>\varphi_i(s_i(\xi_i''))-\psi_i(\xi_i's_i(\xi_i'')-\varphi_i(s_i(\xi_i''))),$$
$$\varphi_i(s_i(\xi_i'))-\psi_i(\xi_i''s_i(\xi_i')-\varphi_i(s_i(\xi_i')))<$$
$$<\varphi_i(s_i(\xi_i''))-\varphi_i(\xi_i''s_i(\xi_i'')-\varphi_i(s_i(\xi_i''))),$$

whence

$$\psi_i(\xi_i''s_i(\xi_i')-\varphi_i(s_i(\xi_i')))-\psi_i(\xi_i's_i(\xi_i')-\varphi_i(s_i(\xi_i')))>$$
$$>\psi_i(\xi_i''s_i(\xi_i'')-\varphi_i(s_i(\xi_i'')))-\psi_i(\xi_i's_i(\xi_i'')-\varphi_i(s_i(\xi_i''))). \tag{A.3}$$

Consider two cases.

1. Let $\xi_i's_i(\xi_i')>\varphi_i(s_i(\xi_i'))$. Then using assumption (A.2) and the properties of the function $\varphi_i(s_i)$, we can easily show that $\xi_i''s_i(\xi_i')>\varphi_i(s_i(\xi_i')), \xi_i's_i(\xi_i'')>\varphi_i(s_i(\xi_i'')), \xi_i''s_i(\xi_i'')>\varphi_i(s_i(\xi_i''))$, and therefore

$$\alpha(\xi_i''s_i(\xi_i')-\varphi_i(s_i(\xi_i')))-\alpha(\xi_i's_i(\xi_i')-\varphi_i(s_i(\xi_i')))>$$
$$>\alpha(\xi_i''s_i(\xi_i'')-\varphi_i(s_i(\xi_i'')))-\alpha(\xi_i's_i(\xi_i'')-\varphi_i(s_i(\xi_i''))).$$

We finally obtain the inequality

$$\alpha(\xi_i''-\xi_i')(s_i(\xi_i')-s_i(\xi_i''))>0, \tag{A.4}$$

which contradicts (A.2).

2. If $\xi_i's_i(\xi_i')=\varphi_i(s_i(\xi_i'))$, then $\xi_i''s_i(\xi_i')>\varphi_i(s_i(\xi_i')), \xi_i's_i(\xi_i'')>\varphi_i(s_i(\xi_i'')), \xi_i''s_i(\xi_i'')>\varphi_i(s_i(\xi_i''))$, and similarly to case 1 we obtain (A.4), which contradicts (A.2). Q.E.D.

Proof of Theorem 1. We will first show that there exists at least one winner such that

$$\Delta_i\geqslant s_i(\tilde{\xi}_{i_n})-s_i(\xi_{i_n}^{\max}). \tag{A.5}$$

Assume that this is not so, i.e., for all $i\in Q$, let

$$\Delta_i<s_i(\tilde{\xi}_{i_n})-s_i(\xi_{i_n}^{\max}).$$

Then

$$\sum_{i\in Q}\Delta_i<\sum_{i\in Q}[s_i(\tilde{\xi}_{i_n})-s_i(\xi_{i_n}^{\max})]. \tag{A.6}$$

On the other hand (by (4) and (5)),

$$\sum_{i\in Q}[s_i(\tilde{\xi}_{i_n})-s_i(\xi_{i_n}^{\max})]=c_{\max}-\tilde{c}.$$

We thus have the inequality

$$\sum_{i\in Q}\Delta_i<c_{\max}-\tilde{c},$$

which contradicts (6). Thus, (A.5) holds for at least one winner.

Let us now write out the payoff of this winner before and after coalition formation:

$$\overline{W}_i=\varphi_i(s_i(\xi_{i_n}^{\max}))-\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max}))),$$
$$\tilde{W}_i=\varphi_i(s_i(\tilde{\xi}_{i_n})-\Delta_i)-\psi_i(\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})-\varphi_i(s_i(\tilde{\xi}_{i_n})-\Delta_i)).$$

From these expressions (using (A.5)) we see that it is disadvantageous for winner i to join the coalition if

$$\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max})))<\psi_i(\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})-\varphi_i(s_i(\tilde{\xi}_{i_n})-\Delta_i)). \tag{A.7}$$

We will first show that the function $\xi_i s_i(\xi_i)$ is decreasing in ξ_i. Consider arbitrary ξ_i' and ξ_i'', $\xi_i'>\xi_i''$. By our lemma, $s_i(\xi_i')<s_i(\xi_i'')$, and by (A.1)

$$\xi_i s_i(\xi_i)\geqslant\varphi_i(s_i(\xi_i)).$$

1. Let $\xi_i's_i(\xi_i')=\varphi_i(s_i(\xi_i'))$, then

$$\xi_i''s_i(\xi_i'')\geqslant\varphi_i(s_i(\xi_i''))>\varphi_i(s_i(\xi_i')),$$

i.e., $\xi_i's_i(\xi_i')<\xi_i''s_i(\xi_i'')$.

2. Let $\xi_i's_i(\xi_i')>\varphi_i(s_i(\xi_i'))$ and $\xi_i''s_i(\xi_i'')>\varphi_i(s_i(\xi_i''))$. Since the function $\varphi_i(s_i)$ is continuously differentiable, we have

$$\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i')}=\frac{\alpha}{1+\alpha}\xi_i',$$

$$\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i'')}=\frac{\alpha}{1+\alpha}\xi_i'',$$

and in order to satisfy the inequality $\xi_i's_i(\xi_i') < \xi_i''s_i(\xi_i'')$, we need to have

$$\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i'')}\Big/\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i')}>\frac{s_i(\xi_i')}{s_i(\xi_i'')}. \tag{A.8}$$

Since

$$\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i'')}\leqslant\frac{\varphi_i(s_i(\xi_i''))}{s_i(\xi_i'')},$$

$$\left.\frac{d\varphi_i}{ds_i}\right|_{s_i(\xi_i')}\geqslant\frac{\varphi_i(s_i(\xi_i''))-\varphi_i(s_i(\xi_i'))}{s_i(\xi_i'')-s_i(\xi_i')},$$

$$\frac{\varphi_i(s_i(\xi_i''))-\varphi_i(s_i(\xi_i'))}{s_i(\xi_i'')-s_i(\xi_i')}\leqslant\frac{\varphi_i(s_i(\xi_i''))}{s_i(\xi_i'')}<\frac{\varphi_i(s_i(\xi_i''))}{s_i(\xi_i')},$$

inequality (A.8) holds and we thus have $\xi_i's_i(\xi_i') < \xi_i''s_i(\xi_i'')$.

3. Let $\xi_i's_i(\xi_i')>\varphi_i(s_i(\xi_i'))$ and $\xi_i''s_i(\xi_i'')=\varphi_i(s_i(\xi_i''))$. Define $\bar{s}_i(\xi_i'')$ from the condition

$$\bar{s}_i(\xi_i'')\in\operatorname{Arg}\max_{s_i}[(1+\alpha)\varphi_i(s_i)-\alpha\xi_i''s_i].$$

Clearly, $\bar{s}_i(\xi_i'')\leqslant s_i(\xi_i'')$, and therefore

$$\xi_i's_i(\xi_i')<\xi_i''\bar{s}_i(\xi_i'')\leqslant\xi_i''s_i(\xi_i'').$$

Thus, since $\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})<\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})$, $\varphi_i(s_i(\xi_{i_n}^{\max}))\geqslant\varphi_i(s_i(\tilde{\xi}_{i_n})-\Delta_i)$, inequality (A.7) holds. Q.E.D.

Proof of Theorem 2. Write out the value of the objective function of the center before the formation of the coalition $\bar{W}_c$ and after its formation $\tilde{W}_c$:

$$\bar{W}_c=\sum_{i\in Q}\varphi_i(s_i(\xi_{i_n}^{\max}))+\varphi_{i_n}(c_{\max}),$$

$$\tilde{W}_c=\sum_{i\in Q}\varphi_i(s_i(\tilde{\xi}_{i_n}))+\varphi_{i_n}(\tilde{c}).$$

If "strong" penalties are imposed in the system, then $\psi_i(\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})-\varphi_i(s_i(\tilde{\xi}_{i_n})))=0$, $\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max})))=0$, and therefore using the inequality $\sum_{i\in Q}\Delta_i\geqslant\varphi_{i_n}(c_{\max})-\varphi_{i_n}(\tilde{c})$, we obtain

$$\tilde{W}_c-\bar{W}_c=\sum_{i\in Q}[\varphi_i(s_i(\tilde{\xi}_{i_n}))-\varphi_i(s_i(\xi_{i_n}^{\max}))]-[\varphi_{i_n}(c_{\max})-\varphi_{i_n}(\tilde{c})]\geqslant$$

$$\geqslant\sum_{i\in Q}[\varphi_i(s_i(\tilde{\xi}_{i_n}))-\varphi_i(s_i(\xi_{i_n}^{\max}))]-\sum_{i\in Q}\Delta_i=\sum_{i\in Q}[\tilde{W}_i-\bar{W}_i].$$

Since by assumption a coalition is formed between the winners and the loser only if $i\in Q$ for all $\tilde{W}_i\geqslant\bar{W}_i$, then we finally get

$$\tilde{W}_c\geqslant\bar{W}_c.$$

To prove the theorem in the "weak" penalty case, it suffices to show that

$$\psi_i(\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})-\varphi_i(s_i(\tilde{\xi}_{i_n})))\geqslant\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max}))).$$

If, $\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max})))=0$, then this inequality is obvious. Let $\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i\cdot$

$(\xi_{i_n}^{\max})))>0$. We have to show that the function $\alpha(\xi_i s_i(\xi_i)-\varphi_i(s_i(\xi_i)))$ is increasing in $s_i(\xi_i)$ (or, equivalently, decreasing in ξ_i).

Since

$$\xi_i=\frac{1+\alpha}{\alpha}\varphi_i'(s_i)\,|_{s_i=s_i(\xi_i)},$$

we have

$$\alpha(\xi_i s_i(\xi_i)-\varphi_i(s_i(\xi_i)))=(1+\alpha)s_i(\xi_i)\varphi_i'(s_i(\xi_i))-\alpha\varphi_i(s_i(\xi_i)).$$

The condition that the function $\alpha(\xi_i s_i(\xi_i)-\varphi_i(s_i(\xi_i)))$ is increasing thus has the form

$$(1+\alpha)s_i(\xi_i)\varphi_i''(s_i(\xi_i))+\varphi_i'(s_i(\xi_i))\geqslant 0.$$

Using this inequality, we obtain

$$\tilde{W}_c-\overline{W}_c=\sum_{i\in Q}[\varphi_i(s_i(\tilde{\xi}_{i_n}))-\varphi_i(s_i(\xi_{i_n}^{\max}))]-[\varphi_{i_n}(c_{\max})-\varphi_{i_n}(\tilde{c})]\geqslant$$
$$\geqslant\sum_{i\in Q}[\varphi_i(s_i(\tilde{\xi}_{i_n}))-\psi_i(\tilde{\xi}_{i_n}s_i(\tilde{\xi}_{i_n})-\varphi_i(s_i(\tilde{\xi}_{i_n})))-\varphi_i(s_i(\xi_{i_n}^{\max}))+$$
$$+\psi_i(\xi_{i_n}^{\max}s_i(\xi_{i_n}^{\max})-\varphi_i(s_i(\xi_{i_n}^{\max})))]-\sum_{i\in Q}\Delta_i=\sum_{i\in Q}[\tilde{W}_i-\overline{W}_i].$$

Finally,

$$\tilde{W}_c\geqslant\overline{W}_c,$$

Q.E.D.

Let us now derive an estimate of the set of feasible values of α.

The inequality $\alpha\geqslant 0$ holds by assumption.

From the inequality $(1+\alpha)s_i\varphi_i''(s_i)+\varphi_i'(s_i)\geqslant 0$, we have

$$\alpha\leqslant-\frac{s_i\varphi_i''(s_i)+\varphi_i'(s_i)}{s_i\varphi_i''(s_i)}.$$

Since $s_i(\xi_{i_n}^{\max})\leqslant s_i\leqslant s_i(\xi_{i_n}^{\min})$ during the operation of the system, then $\alpha\leqslant\alpha_U$.

The inequality $\alpha_U > 0$ follows from the inequalities

$$s_i\varphi_i''(s_i)<0,\quad s_i\varphi_i''(s_i)+\varphi_i'(s_i)=\frac{\alpha}{1+\alpha}[\xi_i s_i(\xi_i)]_{s_i(\xi_i)}>0$$

(see proof of Theorem 1). This completes the proof of the theorem and the corollary.

LITERATURE CITED

1. V. N. Burkov, B. Danev, A. K. Enaleev, et al., "Competition mechanisms for allocation of scarce resources," Avtomat. Telemekh., No. 11, 142-153 (1988).