CORRECT ADAPTIVE MECHANISMS

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We consider an active system comprising a center and a far-sighted active element with a controlled stochastic model of constraints. The notion of correctness of adaptive functioning mechanisms (AFMs) is generalized under conditions of uncertainty (including the case of unrealistic plans). Optimal synthesis of a correct AFM is considered. The characteristic function of the AFM yielding the necessary and sufficient conditions of correctness is constructed. The optimal incentive procedure is synthesized for a correct AFM. Open and two-way planning problems in AFM are formulated and solved. The results are applied to the case of guaranteed correct AFMs.

INTRODUCTION

Optimal synthesis of a progressive AFM of an active system ensuring full utilization of the system potential was considered in [1]. The active system included a center and a far-sighted active element (AE) with a controlled stochastic model of constraints. The AFM characteristic was introduced and used to derive the necessary and sufficient conditions of optimality for a progressive AFM. The characteristic also provided the basis for deriving constructive necessary and sufficient conditions of optimality of the incentive procedure for a progressive AFM with given prediction and planning procedures. The initial problem of optimal synthesis of a progressive AFM for a vector AE was reduced to the construction of a progressive AFM for some scalar AE.

Optimal synthesis of a guaranteed correct AFM, i.e., an AFM which ensures fulfillment of a plan that is realizable for any active-system potential, was considered in [2]. Such a plan corresponds to the least potential, and a guaranteed correct AFM accordingly ensures minimal utilization of the system potential. It is therefore relevant for both theory and practice to investigate and construct AFMs that motivate the AEs to fulfill the plan and at the same time to achieve full utilization of their potential.

In this paper, we consider the important and quite common problem of constructing a correct AFM that ensures fulfillment of the centrally imposed plan by the active elements in cases when the planned targets are attainable in the prevailing situation (i.e., for the specific values of the random factors of the potential, which are unknown to the center). Alternatively, with an unrealistic plan, a correct AFM should motivate the AEs to increase the degree of plan fulfillment to the maximum attainable level in the given situation through complete utilization of the system potential.

2. THE MODEL

The set of possible states of the AEs, the AE objective function, the order of functioning of the system, and the solution of the game have the same form as in [1]. We accordingly use the same notation as in [1]. Recall that the state of an AE in period t is described by the vector $\mathbf{y}_t=(y_{1t},\ldots,y_{mt})\in Y(\mathbf{p}_t)$ where $\mathbf{p}_t=(a_t\zeta_t)$ is the vector parameter of the model of constaints of the AE (the model of the AE potential), ζ_t is a random disturbance, $\zeta_t\in\theta$, a_t is a tunable parameter estimated by the prediction procedure I:

$$a_{t+1}=I(a_t,\mathbf{y}_t)\uparrow a_t,\ a_0=a^0. \tag{1}$$

Here and in what follows, we use the notation $I(a_t,\mathbf{y}_t)\uparrow a_t$ if $I(a_t,\mathbf{y}_t)$ is strictly monotone increasing in a_t. It is easy to see that $a_t\in A_t$, $\mathbf{p}_t\in P_t\equiv A_t\otimes\theta$ where $A_t\otimes\theta$ is the direct product of A_t and θ, $A_t\otimes Y_t\underset{I}{\rightarrow}A_{t+1}$, $A_0=A^0$, $Y_t=Y(p_t)$. The set $Y(a_t,\zeta_t)$ is convex, compact, strictly

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monotone and Hausdorff continuous on A_t for any $\zeta_t \in \theta$ and on θ for any $a_t \in A_t$, $Y(a_t, \zeta_t) \supset Y(a_t, \zeta_t')$, $\zeta_t > \zeta_t'$, $\zeta_t, \zeta_t' \in \theta$; $Y(\mathbf{p}_t) \supset W(\mathbf{p}_t)$ where $W(\mathbf{p}_t)$ is the boundary of $Y(\mathbf{p}_t)$. We assume that $A_{t+1} \subset A_t$, $t = 0, 1, \ldots$, and for any $a \in A_t$, $\zeta, \zeta' \in \theta$,

$$I(a, \mathbf{y}) \geqslant I(a, \mathbf{y}') \Leftrightarrow \exists \zeta, \zeta' \in \theta\colon \mathbf{y} \in W(a, \zeta),\ \mathbf{y}' \in W(a, \zeta'),\ \zeta \geqslant \zeta', \tag{2}$$

where the equality holds if and only if $\zeta = \zeta'$. Hence

$$I(a, \mathbf{y}) > I(a, \mathbf{y}'),\ \mathbf{y} \in W(a, \zeta),\ \mathbf{y}' \in \operatorname{int} Y(a, \zeta). \tag{3}$$

The AE objective function is

$$w(\varphi_t, \ldots, \varphi_{t+T}) = \sum_{\tau=t}^{t+T} \rho^{t-\tau}\varphi_\tau,\ \varphi_\tau = f(\mathbf{x}_\tau, \mathbf{y}_\tau),\ \mathbf{x}_\tau = \pi(a_\tau), \tag{4}$$

where φ_τ is the reward in period τ, ρ is the discounting factor, $\rho \leq 1$, $\mathbf{x}_\tau$ is the plan in period τ, $\mathbf{x}_\tau \in X_\tau = \{\mathbf{x} = \pi(a), a \in A_\tau\}$, $\pi(\cdot)$, $f(\cdot)$ are continuous functions of their arguments.

We consider the usual order of functioning of the system in period t [1]. In period t, the center estimates a_t (1) and assigns the plane $\mathbf{x}_t$ (4). Then, realizing the random potential $\mathbf{p}_t$ the AE chooses the state $\mathbf{y}_t$ and receives the reward φ_t. In the process, the AE optimizes the objective function (4) for some predicted values of its potential and state over the entire far-sighted interval T. Since the choice of the state $\mathbf{y}_\tau$ (for a given potential $\mathbf{p}_\tau$ depends on the AE, the predictor states are the states maximizing the objective function (4). We introduce a maximization operator over the states $\mathbf{y}_\tau$ from the set $Y(\mathbf{p}_\tau)$: $M_\tau = \max\limits_{y_\tau \in Y(p_\tau)}$. By $E\zeta_t$ we denote the operator that eliminates the uncertainly concerning the disturbance ζ_t in period τ, $\zeta_\tau \in \theta$. For example, if uncertainty is eliminated using the principle of maximum guaranteed outcome, $E_{\zeta_\tau} = \min\limits_{\zeta_\tau \in \theta}$. The product of the operators E_{ζ_t}, $\tau = \overline{\nu, \mu}$ will be denoted by $E_\nu{}^\mu$: $E_\nu{}^\mu = E_{\zeta_\nu} \ldots E_{\zeta_\mu}$. Finally, the sequential product of the operators M_τ and E_{ζ_t}, $\tau = \overline{\nu, \mu}$ will be denoted by $M_\nu{}^\mu$: $M_\nu{}^\mu = E_{\zeta_\nu} M_\nu \ldots E_{\zeta_\mu} M_\mu$. The expected AE payoff in state $\mathbf{y}_t$ thus can be written in the form [1]

$$\hat{w}(\mathbf{x}_t, \mathbf{y}_t) = M_{t+1}^{t+T} w(\varphi_t, \ldots, \varphi_{t+T}) = \varphi_t + \sum_{\tau=t+1}^{t+T} \rho^{\tau-t} M_{t+1}^{t+T} \varphi_\tau(\mathbf{x}_t, \mathbf{y}_t), \tag{5}$$

where $\varphi_\tau(\mathbf{x}_t, \mathbf{y}_t) \equiv f(\mathbf{x}_\tau, \mathbf{y}_\tau)$, $\mathbf{x}_\tau = \pi(a_\tau)$, $a_{\tau+1} = I(a_\tau, \mathbf{y}_\tau)$, $\mathbf{y}_\tau \in Y(\mathbf{p}_\tau)$, $\mathbf{p}_\tau \in P$, $\tau = \overline{t, t+T}$.

The set of possible choices of the AE (the solution of the game) is given by

$$R(\Sigma, \mathbf{p}) = \operatorname*{Arg\,max}_{\mathbf{y}_t \in Y(\mathbf{p})} \hat{w}(\mathbf{x}_t, \mathbf{y}_t). \tag{6}$$

3. A CORRECT AFM

As we have noted in the introduction, a guaranteed correct AFM is an AFM that ensures equality of the AE state and plan ($\mathbf{y}_t = \mathbf{x}_t \in Y(\mathbf{p}_t)$) for any system potential ($R(\Sigma, \mathbf{p}_t) = \mathbf{x}_t\ \forall \mathbf{p}_t \in P$). Note that the plane $\mathbf{x}_t$ is assigned before the element determines the disturbance ζ_t and with it the system potential $\mathbf{p}_t = (a_t, \zeta_t)$. Moreover, the disturbance ζ_t is not known to the center. Therefore the plan $\mathbf{x}_t$ does not necessarily belong to the set of possible AE states $Y(\mathbf{p}_t)$. We say that the plan $\mathbf{x}_t$ is realistic (unrealistic) if $\mathbf{x}_t \in Y(\mathbf{p}_t)$ (or respectively $\mathbf{x}_t \overline{\in} Y(\mathbf{p}_t)$. The definition of a guaranteed correct AFM is obviously meaningful only if the plan is realistic. We generalize the definition of a correct AFM to the case of an unrealistic plan. We assume that in this case a correct AFM ensures that the elements choose a state $\mathbf{y}_t$ from some subset $A(\mathbf{x}_t, \mathbf{p}_t)$ on the boundary of the set of possible states $W(\mathbf{p}_t) \supset A(\mathbf{x}_t, \mathbf{p}_t)$ which in a certain sense is "the closest" to the desired state $\mathbf{x}_t$ [1]. Let

$$B(\mathbf{x}_t, \mathbf{p}_t) = \begin{cases} \mathbf{x}_t, & \mathbf{x}_t \in Y(\mathbf{p}_t) \\ A(\mathbf{x}_t, \mathbf{p}_t) \subset W(\mathbf{p}_t), & \mathbf{x}_t \overline{\in} Y(\mathbf{p}_t) \end{cases},\ \mathbf{p}_t \in P \tag{7}$$

be the set of optimal AE states in period t, t = 0, 1, ..., and $A(\mathbf{x}_t, \mathbf{p}_t^x) = \mathbf{x}_t$ if $\mathbf{x}_t \in W(\mathbf{p}_t^x)$. We say that the AFM is correct if

$$R(\Sigma, \mathbf{p}_t) \subset B(\mathbf{x}_t, \mathbf{p}_t) \ \forall \mathbf{p}_t \in P. \tag{8}$$

In the absence of far-sightedness (T = 0), the sum in the right-hand side of (5) is zero and condition (8), by (5)-(7), has the form $\forall \mathbf{p}_t \in P$

$$F(\mathbf{x}_t, \mathbf{p}_t) = \mathrm{Arg} \max_{\mathbf{y}_t \in Y(\mathbf{p}_t)} | \ f(\mathbf{x}_t, \mathbf{y}_t) \subset B(\mathbf{x}_t, \mathbf{p}_t). \tag{9}$$

The class of AFMs satisfying (9) will be denoted by Π. It is natural to restrict the construction of correct AFMs to mechanisms from the class Π, because the AFMs obtained in this way remain correct for AEs that are not far-sighted. Finally, recall that the AFM Σ = (I, π, f) is called progressive (regressive) if $M_t f(\pi(a_t), \mathbf{y}_t)$ increases (respectively, decreases) in a a_t for any $\mathbf{p}_t \in P$, t = 0, 1, ... [1]. Progressive and regressive AFMs comprise the class M of monotone AFMs. The solution of the synthesis problem of a correct AFM (8) will be sought in the class (Π ∩ M). We first assume that $A(\mathbf{x}_t, \mathbf{p}_t)$ is a point for any $\mathbf{p}_t \in P$.

THEOREM 1. Let $\Sigma \in (\Pi \cap M)$. Then the AFM Σ is correct if and only if for any $\mathbf{p}_t \in P$ we have

$$V(\Sigma, \mathbf{p}_t) \subset B(\mathbf{x}_t, \mathbf{p}_t), \tag{10}$$

where

$$V(\Sigma, \mathbf{p}_t) = \mathrm{Arg} \max_{\mathbf{y}_t \in Y(\mathbf{p}_t)} v(\mathbf{x}_t, \mathbf{y}_t), \tag{11}$$

$$v(\mathbf{x}_t, \mathbf{y}_t) = f(\mathbf{x}_t, \mathbf{y}_t) + \sum_{\tau=t+1}^{t+T} \rho^{\tau-t} E_{t+1}^{t+T} \tilde{\varphi}_\tau(\mathbf{x}_t, \mathbf{y}_t), \tag{12}$$

$$\tilde{\varphi}_\tau(\mathbf{x}_t, \mathbf{y}_t) = f(\tilde{\mathbf{x}}_\tau, \mathbf{z}_\tau), \ \mathbf{z}_\tau \in B(\tilde{\mathbf{x}}_\tau, \tilde{\mathbf{p}}_\tau), \ \tilde{\mathbf{x}}_\tau = \pi(\tilde{a}_\tau), \ \tilde{\mathbf{p}}_\tau = (\tilde{a}_\tau, \zeta_\tau),$$
$$\tilde{a}_\tau = I(\tilde{a}_{\tau-1}, \mathbf{z}_\tau), \ \tau = t+1, t+T, \ \tilde{a}_t = a_t, \ \mathbf{z}_t = \mathbf{y}_t$$

and

$$R(\Sigma, \mathbf{p}_t) = V(\Sigma, \mathbf{p}_t) = F(\mathbf{x}_t, \mathbf{p}_t).$$

This and the following theorems are proved in the Appendix.

Note that for $\mathbf{x}_t \overline{\in} \bigcup_{\mathbf{p} \in P} Y(\mathbf{p})$, t = 0, 1, ... (unrealistic plane) by (7) we have $B(\mathbf{x}_t, \mathbf{p}_t) = A(\mathbf{x}_t, \mathbf{p}_t)$ and Theorem 1 leads to Theorem 2 [1]. Expression (12) then coincides with (18) [1]. Therefore (12) can be considered as a generalization of the AFM characteristic which includes the case of both progressive and correct AFM. Now, the AFM Σ satisfying Theorem 1 is correct both for AEs without far-sightedness T = 0, Σ ∈ (Π ∩ M), and for far-sighted AEs with T > 0. It is easy to show that it is also correct for AEs with far-sightedness less than T, i.e., far-sightedness equal to 1, 2, ..., T − 1. Thus, if the true far-sightedness of an AE t_{tr} is not known exactly to the center, it suffices to take T that is definitely less than T_{tr} and to ensure that the conditions of Theorem 1 are satisfied.

4. CORRECT INCENTIVES

Let us now construct incentive procedures that ensure correctness of AFM for given prediction procedure I and planning procedure π. As we have noted above, the solution of the optimal synthesis problem (8) obtained in [1] remains valid for the case $\mathbf{x}_t \overline{\in} \bigcup_{\mathbf{p} \in P} Y(\mathbf{p})$. Therefore, in what follows we seek the solution in the case of AFMs Σ ∈ R with procedures I, π such that $x_t \in \bigcup_{\mathbf{p} \in P} Y(\mathbf{p})$ (i.e, the plan is realistic for at least some value of the AE potential). We introduce an extension of the set $B(\mathbf{x}_t, \mathbf{p}_t)$ in the form

$$B_+(\mathbf{x}_t, \mathbf{p}_t) = \begin{cases} B(\mathbf{x}_t, \mathbf{p}_t), & \mathbf{x}_t \in \mathrm{int}\, Y(\mathbf{p}_t), \\ \mathscr{E}(\mathbf{x}_t, \mathbf{p}_t), & \mathbf{x}_t \in \mathrm{int}\, Y(\mathbf{p}_t), \end{cases} \tag{13}$$

where $\mathscr{E}(\mathbf{x}_t, \mathbf{p}_t)$ is a closed set, $\mathscr{E}(\mathbf{x}_t, \mathbf{p}_t) \subset W(\mathbf{p}_t)$ and $\mathscr{E}(\mathbf{x}_t, \mathbf{p}_t^x) = \mathbf{x}_t$ if $\mathbf{x}_t \in W(\mathbf{p}_t^x)$.

THEOREM 2. Consider the AFM $\Sigma = (I, \pi, f) \in (\Pi \cap M \cap R)$. The AFM Σ is correct if and only if for any $\mathbf{p}_t \in P$ we have

$$f(\mathbf{x}_t, \mathbf{y}_t) = h(a_t, a_{t+1}) \quad \forall \mathbf{y}_t \in B_+(\mathbf{x}_t, \mathbf{p}_t), \tag{14}$$

where h is a continuous function of its arguments, and

$$h(a_t, a_{t+1}) \uparrow a_{t+1}, u(a_t, a_{t+1}) \uparrow a_{t+1}, a_{t+1} \leqslant a^*_{t+1}, \tag{15}$$

$$\arg \max_{a_{t+1} \in A_{t+1}} h(a_t, a_{t+1}) = \arg \max_{a_{t+1} \in A_{t+1}} u(a_t, a_{t+1}) = \min [a^*_{t+1}, I(a_t, \mathbf{y}_t)], \tag{16}$$

where

$$u(a_t, a_{t+1}) = \sum_{\tau=t}^{t+T} \rho^{\tau-t} E_{t+1}^{t+T} h(\tilde{a}_\tau, \tilde{a}_{\tau+1}), \tilde{a}_t = a_t, \tilde{a}_{t+1} = a_{t+1}, \tilde{a}_{\tau+1} = I(\tilde{a}_\tau, \mathbf{z}_\tau), z_\tau \in B(\tilde{\mathbf{x}}_\tau, \mathbf{p}_\tau), \tilde{\mathbf{p}}_\tau = (\tilde{a}_\tau, \zeta_\tau), \tau = \overline{t+1, t+T-1}, \quad a^*_{t+1} = I(a_t, \pi(a_t)). \tag{17}$$

Theorem 2 has the following meaning. A correct incentive procedure is realized by the function $f(\mathbf{x}_t, \mathbf{y}_t)$ whose maximum for $\mathbf{y}_t \in W(\mathbf{p}_t)$ (i.e., for a given $a_{t+1} = I(a_t, \mathbf{y}_t)$) is attained, by (9) and (13), on the set $B_+(\mathbf{x}_t, \mathbf{p}_t)$. When $\mathbf{y}_t$ is varied within the sets $B_+(\mathbf{x}_t, \mathbf{p}_t)$ with different $\mathbf{p}_t \in P$ the function $f(\mathbf{x}_t, \mathbf{y}_t) = h(a_t, a_{t+1})$ increases monotonically with the increase of $a_{t+1} = I(a_t, \mathbf{y}_t)$ if $a_{t+1} \le a_{t+1}^*$ (i.e., until the equality $\mathbf{x}_t = \mathbf{y}_t$ is attained). By (16), the maximum of $f(\mathbf{x}_t, \mathbf{y}_t)$ is attained either for $\mathbf{x}_t = \mathbf{y}_t$ or for $\mathbf{y}_t = \mathbf{z}_t \in A(\mathbf{x}_t, \mathbf{p}_t)$ (if $I(a_t, \mathbf{z}_t) \leqslant a^*_{t+1}$). Conditions (15)-(17) use the characteristic of the incentive procedure $u(a_t, a_{t+1})$ to tune the parameters of the function $f(\mathbf{x}_t, \mathbf{y}_t)$ for far-sighted AEs with given prediction and planning procedures. Thus, the incentive procedure in a correct AFM, as in a progressive AFM [1], is based on scalar estimates a_{t+1} as the images of the vector AE states $\mathbf{y}_t$.

Note that by (13) the set of maxima of $f(\mathbf{x}_t, \mathbf{y}_t)$ for $\mathbf{x}_t \in \operatorname{int} Y(\mathbf{p}_t)$, $\mathbf{y}_t \in W(\mathbf{p}_t)$ is arbitrary (because $\mathscr{E}(\mathbf{x}_t, \mathbf{p}_t) \subset W(\mathbf{p}_t)$ is arbitrary). Therefore condition (13) essentially does not restrict the choice of the functions $f(\cdot)$ when synthesizing correct incentive systems in AFMs. It is only essential that for $\mathbf{y}_t \in W(\mathbf{p}_t)$ the values of $f(\mathbf{x}_t, \mathbf{y}_t) = h(a_t, a_{t+1})$ and $u(a_t, a_{t+1})$ be less than $h(a_t, a^*_{t+1})$ and $u(a_t, a^*_{t+1})$. Otherwise, a far-sighted AE will find it advantageous to exceed the plan $\mathbf{x}_t$.

5. GUARANTEED CORRECT MECHANICS

Let $G \subset R$ be the class of AFMs with the procedures I, π such that $\mathbf{x}_\tau \in Y(\mathbf{p}_\tau)$, $\tau = \overline{t, t+T}$, $\mathbf{p}_t \in P$ (i.e., all the plans are realistic). To this end it suffices to have $\mathbf{x}_\tau \in \bigcap_{\mathbf{p}_\tau \in P} Y(\mathbf{p}_\tau)$. Represent the incentive procedure in the form

$$f(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}) - \chi(\mathbf{x}, \mathbf{y}), \chi(\mathbf{x}, \mathbf{x}) = 0, \chi(\mathbf{x}, \mathbf{y}) > 0, \mathbf{x} \neq \mathbf{y}, \tag{18}$$

where $g(\mathbf{y})$ is a continuous function. Let $\Sigma \in (\Pi \cap M \cap G)$. Then from (18) it follows that the function $g(\pi(a))$ is monotone in a, $a \in A$. Indeed, for $\Sigma \in (G \cap \Pi)$ we have from (7), (9) $M_\tau f(\pi(a_\tau), \mathbf{y}_\tau) = g(\pi(a_\tau))$ and from $\Sigma \in M$ it follows that $g(\pi(a_\tau))$ is monotone in a, $a \in A$. Then, substituting (18) in the condition of Theorem 1, we obtain that for an AFM to be guaranteed correct (see Sec. 3) it is necessary and sufficient that for any $\mathbf{p}_t \in P$ we have

$$\arg \max_{\mathbf{y}_t \in Y(\mathbf{p}_t)} \left[f(\mathbf{x}_t, \mathbf{y}_t) + \sum_{\tau=t+1}^{t+T} \rho^{\tau-t} g(\tilde{\mathbf{x}}_\tau) \right] = \mathbf{x}_t \in Y(\mathbf{p}_t), \tag{19}$$

where $\tilde{\mathbf{x}}_\tau = \pi(\tilde{a}_\tau)$, $\tilde{a}_{\tau+1} = I(\tilde{a}_\tau, \tilde{\mathbf{x}}_\tau)$, $\tau = \overline{t+1, t+T-1}$, $\tilde{a}_{t+1} = I(a_t, \mathbf{y}_t)$. Condition (19) establishes an addition lower bound on the penalty function $\chi(\mathbf{x}_t, \mathbf{y}_t)$. Indeed, from (9), (18), (19) we have

$$\chi(\mathbf{x}_t, \mathbf{y}_t) > g(\mathbf{y}_t) - g(\mathbf{x}_t) + \psi(\mathbf{x}_t, \mathbf{y}_t), \mathbf{x}_t \neq \mathbf{y}_t, \tag{20}$$

where $\Psi(\mathbf{x}_t, \mathbf{y}_t) = \max \left[0, \sum_{\tau=t+1}^{t+T} (g(\tilde{\mathbf{x}}_\tau) - g(\hat{\mathbf{x}}_\tau))\right]$, $\hat{\mathbf{x}}_\tau = \pi(\hat{a}_\tau)$, $\hat{a}_{\tau+1} = I(\hat{a}_\tau, \hat{\mathbf{x}}_\tau)$, $\tau = \overline{t+1, t+T-1}$, $\hat{a}_{t+1} = I(a_t, \mathbf{x}_t)$, $\tilde{\mathbf{x}}_\tau$ is defined in (19). Note that for $\sum_{\tau=t+1}^{t+T} [g(\tilde{\mathbf{x}}_\tau) - g(\hat{\mathbf{x}}_\tau)] = 0$ (20) coincides with (9), which is a necessary and sufficient condition for a mechanism to be correct without far-sightedness (i.e., when T = 0). Now, if the AFM Σ is progressive, then we easily see that the additional penalities $\psi(\mathbf{x}_t, \mathbf{y}_t)$ associated with far-sightedness are nonzero for $I(a_t, \mathbf{y}_t)>I(a_t, \mathbf{x}_t)$, are equal for any $\mathbf{y}_t \in W(\mathbf{p}_t)$ for given $\mathbf{p}_t \in P$, and increase with the increase of $I(a_t, \mathbf{y}_t)$. In view of (3), this means that the penalties for "exceeding" the plan (i.e., $\mathbf{y}_t \in W(a, \zeta)$, $\mathbf{x}_t \in \operatorname{int} Y(a, \zeta)$) for a far-sighted AE should be greater than for an AE without far-sightedness. If the AFM is regressive, then the additional penalties $\psi(\mathbf{x}_t, \mathbf{y}_t)$ are nonzero for $a_{t+1}=I(a_t, \mathbf{y}_t)<\hat{a}_{t+1}=I(a_t, \mathbf{x}_t)$ and increase with the decrease of $a_{t+1}=I(a_t, \mathbf{y}_t)$. In this case, far-sightedness increases the penalties for "underperformance" (i.e., reduction of a_{t+1}).

6. OPEN AFM

Assume that the plan for period t is assigned by the AE itself (so-called open planning), after which the system functions as described above. The AE essentially sets the initial plan in an iterative plan-forming procedure. Substantively this corresponds to open current planning of the AE states followed by adaptive estimation of the parameters of its model of constraints and long-term planning on the basis of these estimates.

We denote by π^0 a planning procedure that includes open current planning and adaptive long-term planning (4). The functioning mechanism with planning procedure π^0, prediction procedure I (1), and incentive procedure f (4) will be called an "open" AFM. We denote open AFMs by $\Sigma = (I, \pi^0, f)$. In an open AFM, in distinction from an "ordinary" AFM, the element first chooses the plan $\mathbf{x}_t$ and then chooses the state $\mathbf{y}_t$. Optimal open AFMs naturally include also those mechanisms for which the AE plan and the AE state belong respectively to the set of optimal plans and to the set of optimal states. Since the choice $\mathbf{y}_\tau$ of $\tau = t + 1, t + T$ also depends on the AE, the predictors as before are the states maximizing the objective function (4). The expected payoff of the AE under the plan $\mathbf{x}_t$ obviously has the form

$$\hat{w}(\mathbf{x}_t) = M_t^{t+T} w(\varphi_t, \ldots, \varphi_{t+T}), \tag{21}$$

where $\mathbf{x}_t \in X_t$, X_t is the set of feasible plans. Therefore, the set of preferred plans for an AE plan under open planning has the form $Q(\Sigma, X_t) = \operatorname{Arg} \max\limits_{\mathbf{x}_t \in X_t} \hat{w}(\mathbf{x}_t)$.

Denote the set of optimal plans by X_t^0. In view of the above, the open AFM $\Sigma = (I, \pi^0, f)$ is called optimal if $Q(\Sigma, X_t) \subset X_t^0$ and (8) holds.

THEOREM 3. Consider an open AFM $\Sigma = (I, \pi^0, f) \in (\Pi \cap M)$. The AFM Σ is optimal if and only if

$$V(\Sigma, p_t) \subset B(\mathbf{p}_t), \mathbf{p}_t \in P, \tag{22}$$

$$\operatorname{Arg} \max_{\mathbf{x}_t \in X_t} v(\mathbf{x}_t) \subset X_t^0, \tag{23}$$

where $v(\mathbf{x}_t) = E_{\zeta_t} v(\mathbf{x}_t, \mathbf{z}_t), \mathbf{z}_t \in B(\mathbf{p}_t)$.

The function $v(\mathbf{x}_t)$ will be called the characteristic of an open AFM.

7. TWO-WAY AFM

Now assume that in period t the plan $\mathbf{x}_t = \pi(a_t)$ is assigned by the center on the basis of a current estimate of the parameters of the model of AE constraints a_t as reported by the element to the center (the so-called two-way or bottom-up method of data generation), after which the system functions precisely as in the base model. Substantively this corresponds to two-way current planning with subsequent generation of adaptive estimates of AE parameters and long-term planning on the basis of these estimates.

We denote by I^0 the prediction procedure which includes two-way current estimation of the AE parameters and adaptive prediction of these parameters in the long term by means of the procedure I. The functioning mechanism with prediction procedure I^0, planning procedure π, and incentive procedure f (5) will be called a two-way AFM and will be denoted by $\Sigma = (I^0, \pi, f)$. The notion of optimality of two-way AFMs is introduced similarly to the notion of optimality of open AFMs. Denote by A_t and A_t^0 respectively the sets of the center's feasible and optimal estimates a_t, $A_t^0 \subset A_t$. The two-way AFM $\Sigma = (I^0, \pi, f)$ will be called optimal if $\operatorname{Arg} \max\limits_{a_t \in A_t} \hat{w}(\pi(a_t)) \subset A_t^0$ and (8) holds. Then from Theorem 3 we obtain

COROLLARY. Consider the two-way AFM $\Sigma = (I^0, \pi, f) \in (\Pi \cap M)$. Σ is optimal if and only if

$$V(\Sigma, \mathbf{p}_t) \subset B(\mathbf{x}_t, \mathbf{p}_t),\ \mathbf{p}_t \in P,\ \operatorname{Arg}\max_{a_t \in A_t} v(\pi(a_t)) \subset A_t^0.$$

An example is provided by a guaranteed correct two-way AFM. Let $\Sigma \in (\Pi \cap M \cap G)$ and (18) holds. Then condition (22) takes the form (19) and the characteristic function of the open AFM is

$$v(\mathbf{x}_t) = g(\mathbf{x}_t) + \sum_{\tau=t+1}^{t+T} \rho^{\tau-t} g(\tilde{\mathbf{x}}_\tau) = \hat{w}(\mathbf{x}_t),$$

where $\tilde{x}_\tau$ is given by (19). Note that $v(\pi(a_t))$ is monotone in a_t, $a_t \in A_t$, because, first $\Sigma \in M$ and $h(\pi(a_t))$ is monotone in a_t, and, second, $I(a_t, y_t) \uparrow a_t$. Now, if $a_t \in A_t^G = \{a_t | \mathbf{x}_t \in Y(\mathbf{p}_t), \mathbf{x}_\tau \in Y(\mathbf{p}_\tau), \tilde{\mathbf{p}}_\tau = (\tilde{a}_\tau, \zeta_\tau), \tau = \overline{t+1, t+T}\}$ where $\tilde{a}_\tau$ is given by (19), then the two-way AFM Σ is guaranteed correct (i.e., $\Sigma \in G$). Let Σ be progressive, so that $h(\pi(a_+)) \uparrow a_t$. Then, clearly $\arg\max_{a_t \in A_t^G} v(\pi(a_t)) = \max_{a_t \in A_t^G} a_t = a_{t*}$. Thus, the AE will report to the center the highest estimate a_t^* from all the estimates $a_t \in A_t^G$ for which the plans are guaranteed to be realiable in the periods t, ..., t + T. If $a_t^* \in A_t^0$, then, by Corollary, this mechanism is optimal, and otherwise it is not optimal. If the AFM Σ is regressive, then $h(\pi(a_t))$ and $v(\pi(a_t))$ are strictly monotone decreasing in a_t. Therefore the AE reports the minimum estimate $\min_{a_t \in A_t^G} a_t$.

8. CONCLUSIONS

In this paper, we develop an approach to the construction of correct AFMs of far-sighted AEs that are also correct for non-far-sighted AEs (i.e., when $\Sigma \in \Pi$). Using the montonicity property of AFM (i.e., $\Sigma \in M$), we formulate necessary and sufficient conditions for correctness of an AFM in terms of its characteristic function (Theorem 1). We use the characteristic function to solve the synthesis problem of the incentive procedure for realistic† plans ($\Sigma \in R$) and for plans with guaranteed fulfillment ($\Sigma \in G$) (Theorem 2 and Sec. 5). We state the problems of optimal synthesis of AFMs with open planning and with two-way generation of data. These problems are also solved using the AFM characteristic. Thus, the characteristic-based approach to the solution of analysis and synthesis problems of AFMs of active systems with far-sighted elements, as developed in [1] and in the present study, proves to be very fruitful.

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APPENDIX

Proof of Theorem 1. First note that (9) combined with (7) imply that

$$f(\mathbf{x}_\nu, \mathbf{y}_\nu) < f(\mathbf{x}_\nu, \mathbf{x}_\nu), \quad \mathbf{x}_\nu, \mathbf{y}_\nu \in Y(\mathbf{p}_\nu), \quad \mathbf{x}_\nu \neq \mathbf{y}_\nu, \tag{A.1}$$

$$f(\mathbf{x}_\nu, \mathbf{z}_\nu) > f(\mathbf{x}_\nu, \mathbf{y}_\nu), \quad \mathbf{x}_\nu \in Y(\mathbf{p}_\nu), \quad \mathbf{y}_\nu \in Y(\mathbf{p}_\nu)/A(\mathbf{x}_\nu, \mathbf{p}_\nu), \quad \mathbf{z}_\nu \in A(\mathbf{x}_\nu, \mathbf{p}_\nu). \tag{A.2}$$

Introduce the functions

$$v_\nu(\mathbf{x}_\nu, \mathbf{y}_\nu) = \sum_{\tau=\nu+1}^{t+T} \rho^{\tau-\nu} E_{\nu+1}^{t+T} \tilde{\varphi}_\tau(\mathbf{x}_\nu, \mathbf{y}_\nu) + f(\mathbf{x}_\nu, \mathbf{y}_\nu),\ \tilde{\varphi}_\tau(\mathbf{x}_\nu, \mathbf{y}_\nu) = \tag{A.3}$$

$$= f(\mathbf{x}_\tau, \mathbf{z}_\tau),\ \mathbf{x}_\tau = \pi(a_\tau),$$

$$\tilde{a}_\tau = I(\tilde{a}_{\tau-1}, z_{\tau-1}), \quad \mathbf{z}_\tau \in B(\widetilde{\mathbf{x}_\tau}, \tilde{\mathbf{p}}_\tau), \quad \tilde{a}_\nu = a_\nu,$$

$$\tilde{\mathbf{p}}_\tau = (\tilde{a}_\tau, \zeta_\tau) \quad \tau = \overline{\nu+1, \nu+T}, \quad \nu = \overline{t, t+T-1}.$$

Note that by (12) $v_t(\mathbf{x}_t, \mathbf{y}_t) = v(\mathbf{x}_t, \mathbf{y}_t)$ and

$$v_{\nu-1}(\mathbf{x}_{\nu-1}, \mathbf{y}_{\nu-1}) = f(\mathbf{x}_{\nu-1}, \mathbf{y}_{\nu-1}) + \rho E_{\zeta_\nu} v_\nu(\mathbf{x}_\nu, \mathbf{z}_\nu), \quad \mathbf{z}_\nu \in B(\tilde{\mathbf{x}}_\nu, \tilde{\mathbf{p}}_\nu). \tag{A.4}$$

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†Recall that the case of unrealistic plans ($\Sigma \equiv R$) is considered in [1].

From (9) it follows that $f(x_t, z_t)=M_t f(x_t, y_t)$, $z_t \in B(x_t, p_t)$, $x_t \in X_t$, $p_t \in P$. Hence, noting that $A_t \supset A_{t+T}$, $X_t \supset X_{t+T}$, we obtain $f(x_{t+T}, z_{t+T})=M_{t+T} f(x_{t+T}, y_{t+T})$ for any $z_{t+T} \in B(x_{t+T}, p_{t+T})$, $x_{t+T} \in X_{t+T}$, $p_{t+T} \in P$. Now, using the definition (A.1), it is easy to show that

$$v_{t+T-1}(x_{t+T-1}, y_{t+T-1})=f(x_{t+T-1}, y_{t+T-1})+\rho M_{t+T}\varphi_{t+T}(x_{t+T-1}, y_{t+T-1}).$$

The rest of the proof is by induction. Assume that for some ν, $t+1 \le \nu \le t+T-1$, we have

$$v_\nu(x_\nu, y_\nu) = f(x_\nu, y_\nu) + \sum_{\tau=\nu+1}^{t+T} \rho^{\tau-\nu} M_{\nu+1}^{t+T}\varphi_\tau(x_\nu, y_\nu), \tag{A.5}$$

and show that

$$v_{\nu-1}(x_{\nu-1}, y_{\nu-1}) = f(x_{\nu-1}, y_{\nu-1}) + \sum_{\tau=\nu}^{t+T} \rho^{\tau-\nu} M_\nu^{t+T}\varphi_\tau(x_\nu, y_\nu). \tag{A.6}$$

Successively applying the fact that $M_\nu^\mu = E_{\xi_\nu} M_\nu M_{\nu+1}^\mu$ and using the hypothesis (A.5), we obtain

$$\sum_{\tau=\nu}^{t+T} \rho^{\tau-\nu} M_\nu^{t+T}\varphi_\tau(x_{\nu-1}, y_{\nu-1}) = \rho M_\nu v_\nu(x_\nu, y_\nu),$$
$$x_\nu=\pi(a_\nu), \quad a_\nu=I(a_{\nu-1}, y_{\nu-1}), \quad x_{\nu-1}=\pi(a_{\nu-1}), \quad y_\nu \in Y(p_\nu), \quad p_\nu \in P. \tag{A.7}$$

We will first prove necessity and then sufficiency of the conditions of the theorem.

1. Necessity. We will show that for $\Sigma \in (\Pi \cap M)$,

$$M_\nu v_\nu(x_\nu, y_\nu) = v_\nu(x_\nu, z_\nu), \quad z_\nu \in R(\Sigma, p_\nu). \tag{A.8}$$

Note that for $x_\nu \in Y(p_\nu)$, given that Σ is correct, we obtain from (7), (8) $R(\Sigma, p_\nu)=x_\nu$. Hence, using the definition (6), we obtain for $t = \nu$,

$$f(x_\nu, x_\nu) + \sum_{\tau=\nu+1}^{\nu+T} \rho^{\tau-\nu}[M_{\nu+1}^{\nu+T}\varphi_\tau(x_\nu, x_\nu) - M_{\nu+1}^{\nu+T}\varphi_\tau(x_\nu, y_\nu)] > f(x_\nu, y_\nu),$$
$$x_\nu \in X_\nu, \quad y_\nu \in Y(p_\nu), \quad x_\nu \ne y_\nu. \tag{A.9}$$

Now note that if q_ν, $q_\nu \in Y(p_\nu)$ are such that $I(a_\nu, q_\nu) \geqslant I(a_\nu, g_\nu)$ then for the progressive case we obtain

$$\sum_{\tau=\xi+1}^{\mu} \rho^{\tau-\nu}[M_{\nu+1}^\mu\varphi_\tau(x_\nu, q_\nu) - M_{\nu+1}^\mu\varphi_\tau(x_\nu, g_\nu)] \geqslant 0, \tag{A.10}$$

and for the regressive case

$$\sum_{\tau=\xi+1}^{\mu} \rho^{\tau-\nu}[M_{\nu+1}^\mu\varphi_\tau(x_\nu, q_\nu) - M_{\nu+1}^\mu\varphi_\tau(x_\nu, g_\nu)] \leqslant 0 \tag{A.11}$$

for any $\nu \le \xi \le \mu - 1$, $\mu \le \nu + T$.

1.1. Regressive Case. For $x_\nu \bar{\in} Y(p_\nu)$ by (7), $B(x_\nu, p_\nu)=A(x_\nu, p_\nu) \subset W(p_\nu)$. Repeating verbatim the argument used to prove Theorem 2 in [1], we obtain the sought assertion (A.8).

Now let $x_\nu \in Y(p_\nu)$. Assume that $I(a_\nu, x_\nu) \leqslant I(a_\nu, y_\nu)$. Then, setting in (A.11) $\xi=t+T$, $\mu=\nu+T$, $q_\nu=x_\nu$, $g_\nu=y_\nu$, summing the resulting inequality with (A.1), and using (A.5), we obtain (A.8). Now let $I(a_\nu, x_\nu) \geqslant I(a_\nu, y_\nu)$. Setting in (A.11) $\xi=t+T$, $\mu=\nu+T$, $q_\nu=x_\nu$, $g_\nu=y_\nu$ we obtain

$$\sum_{\tau=t+T+1}^{\nu+T} \rho^{\tau-\nu}[M_{\nu+1}^{\nu+T}\varphi_\tau(x_\nu, y_\nu) - M_{\nu+1}^{\nu+T}\varphi_\tau(x_\nu, x_\nu)] \geqslant 0. \tag{A.12}$$

Summing (A.12) and (A.9) and using (A.5), we again obtain (A.8).

1.2. Progressive Case. For $x_\nu \bar{\in} Y(p_\nu)$ by (7), (8), $z_\nu \in A(x_\nu, p_\nu) \subset W(p_\nu)$ so that by (2), (3) $I(a_\nu, z_\nu) \geqslant I(a_\nu, y_\nu)$ if $y_\nu \in Y(p_\nu)/A(x_\nu, p_\nu)$. Hence, setting in (A.10) $\xi=\nu$, $\mu=t+T$, $q_\nu=z_\nu$, $g_\nu=y_\nu$ summing the resulting inequality with (A.2), and using (A.5), we obtain (A.8). Now let $x_\nu \in Y(p_\nu)$. If $I(a_\nu, x_\nu) \geqslant I(a_\nu, y_\nu)$ then setting in (A.10) $\xi=\nu$, $\mu=t+T$, $q_\nu=y_\nu$, $g_\nu=x_\nu$ and subtracting (A.1) from the resulting inequality, we also obtain (A.8). Finally, for $I(a_\nu, x_\nu) \leqslant I(a_\nu, y_\nu)$ setting in (A.10) $\xi = t + T$, $\mu = \nu + T$, $q_\nu=y_\nu$, $g_\nu=x_\nu$. We obtain (A.12). Summing (A.12) and (A.9), we again obtain (A.8).

Thus, (A.8) holds for $\Sigma\in(\Pi\cap M)$. Now, by (8), $z_\nu\in B(x_\nu, p_\nu)$. But then (A.7) and (A.4) imply (A.6) for any $\nu = t+1, t+T$. Setting in (A.6) $\nu = t+1$, noting that $v_t(x_t, y_t)=v(x_t, y_t)$ and comparing (5) and (A.6), we obtain

$$v(x_t, z_t)=\hat{w}(x_t, z_t)>\hat{w}(x_t, y_t)=v(x_t, y_t). \tag{A.13}$$

Therefore, by (6), (11), (12), $V(\Sigma, p_t)=R(\Sigma, p_t)$ and by (8), $z_\nu\in B(x_\nu, p_\nu)$. This completes the proof of necessity.

2. Sufficiency. We will show for $\Sigma\in(\Pi\cap M)$,

$$M_\nu v_\nu(x_\nu, y_\nu)=v_\nu(x_\nu, z_\nu), \quad z_\nu\in V(\Sigma, p_\nu). \tag{A.14}$$

Note that for $x_\nu\in Y(p_\nu)$, $t=\nu$ by (7), (10), we have $V(\Sigma, p_\nu)=x_\nu$. Hence, using the definition (11), we have for $t=\nu$, $x_\nu\neq y_\nu\in Y(p_\nu)$

$$f(x_\nu, x_\nu)+\sum_{\tau=\nu+1}^{\nu+T}\rho^{\tau-\nu}[E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, x_\nu)-E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, y_\nu)]>f(x_\nu, y_\nu). \tag{A.15}$$

Moreover, by (A.1) and (9), $V(\Sigma, p_\nu)=F(x_\nu, p_\nu)$. Now, similarly to (A.10), (A.11), we have for the progressive case

$$\sum_{\tau=\xi+1}^{\mu}\rho^{\tau-\nu}[E_{\nu+1}^{\mu}\tilde{\varphi}_\tau(x_\nu, q_\nu)-E_{\nu+1}^{\mu}\tilde{\varphi}_\tau(x_\nu, g_\nu)]\geqslant 0, \tag{A.16}$$

and for the regressive case

$$\sum_{\tau=\xi+1}^{\mu}\rho^{\tau-\nu}[E_{\nu+1}^{\mu}\tilde{\varphi}_\nu(x_\nu, q_\nu)-E_{\nu+1}^{\mu}\tilde{\varphi}_\tau(x_\nu, g_\nu)]\leqslant 0 \tag{A.17}$$

for any $\nu\le\xi\le\mu\le\nu+T-1$.

2.1. Regressive Case. If $x_\nu\in Y(p_\nu)$ then $B(x_\nu, p_\nu)=A(x_\nu, p_\nu)$ and repeating verbatim the proof of Theorem 2 [1], we obtain (A.14). Now assume that $x_\nu\in Y(p_\nu)$. Then by (7), (8), $B(x_\nu, p_\nu)=x_\nu=z_\nu$ and we have (A.1). If $I(a_\nu, x_\nu)\leqslant I(a_\nu, y_\nu)$ then setting in (A.17) $q_\nu=y_\nu$, $g_\nu=x_\nu$, we obtain

$$\sum_{\tau=\nu+1}^{t+T}\rho^{\tau-\nu}[E_{\nu+1}^{t+T}\tilde{\varphi}_\tau(x_\nu, y_\nu)-E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, x_\nu)]\leqslant 0. \tag{A.18}$$

Summing (A.18) and (A.1) and using (A.5), we obtain (A.14). Now let $I(a_\nu, x_\nu)\geqslant I(a_\nu, y_\nu)$. Then, setting in (A.17) $\xi=t+T$, $\mu=\nu+T$, $q_\nu=x_\nu$, $g_\nu=y_\nu$ we obtain

$$\sum_{\tau=t+T+1}^{\nu+T}\rho^{\tau-t}[E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, y_\nu)-E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, x_\nu)]\geqslant 0. \tag{A.19}$$

Summing (A.15) with (A.19) and using (A.5), we again obtain (A.14).

2.2. Progressive Case. Let $x_\nu\in Y(p_\nu)$. Then by (7), (10), $z_\nu\in A(p_{\nu'}\subset W(p_\nu)$ so that from (2), (3) we obtain $I(a_\nu, z_\nu)\geqslant I(a_\nu, y_\nu)$ if $y_\nu\in Y(p_\nu)/A(p_\nu)$. Hence, setting in (A.16) $\xi=\nu$, $\mu=t+T$, $g_\nu=z_\nu$, $q_\nu=y_\nu$, summing the resulting inequality with (A.2), and using (A.5), we obtain (A.14). Now let $x_\nu\in Y(p_\nu)$. Then by (7), (10), $z_\nu=x_\nu$ and we have (A.9). If $I(a_\nu, x_\nu)\geqslant I(a_\nu, y_\nu)$ then setting in (A.16) $\xi=\nu$, $\mu=t+T$, $g_\nu=x_\nu$, $q_\nu=y_\nu$ and subtracting the resulting inequality from (A.9), we obtain (A.14). Finally, if $I(a_\nu, x_\nu)\leqslant I(a_\nu, y_\nu)$, then setting in (A.16) $\xi=t+T$, $\mu=\nu+T$, $g_\nu=y_\nu$, $q_\nu=x_\nu$ we obtain

$$\sum_{\tau=t+T+1}^{\nu+T}\rho^{\tau-\nu}[E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, y_\nu)-E_{\nu+1}^{\nu+T}\tilde{\varphi}_\tau(x_\nu, x_\nu)]\geqslant 0. \tag{A.20}$$

Now summing (A.20) and (A.15), we again obtain (A.14). Thus, (A.14) holds if $\Sigma\in(\Pi\cap M)$. Now, substituting (A.14) in (A.17) and comparing the resulting expression with (A.4), we obtain the sought equation (A.6) for any ν, $t+1\le\nu\le t+T$. Setting in (A.6) $\nu=t+1$ and using definition (5), we obtain (A.13) for any $x_t\in X_t$, $z_t\in V(\Sigma, p)$, $y_t\in Y(p)/V(\Sigma, p)$. Therefore, by (6), (8), (11), (12), $R(\Sigma, p)=F(x, p)=V(\Sigma, p)\subset B(x, p)$. Q.E.D.

Proof of Theorem 2 is given only for the case of regressive AFM, in order to save space. Fix a_t and $x_t=\pi(a_t)$. Let $\zeta_t\in\theta$ be such that $x_t\in Y(a_t, \zeta_t)$. Then by (2) $a_{t+1}^*=I(a_t, \pi(a_t))>I(a_t, z_t)$,

$p_t=(a_t, \zeta_t)$. Moreover, $B(x_t, p_t)=A(x_t, p_t)$. In this case, the conditions of the theorem (14)- $z_t \in W(p_t)$, (17) fully coincide with conditions (20)-(22) of Theorem 3 [1], whence follow both necessity and sufficiency.

Now consider the case $x_t \in Y(p_t)$. Then $B(x_t, p_t)=x_t \; \forall p_t \in P$ and by (3) $a^*_{t+1} \leqslant I(a_t, z_t)$, $z_t \in W(p_t)$. Let us prove necessity and then sufficiency of the conditions of the theorem.

Necessity. Consider y_t, $y'_t \in \bigcup_{\zeta_t \in \theta} A(x_t, p_t)$ such that $a_{t+1} > a_{t+1} = I(a_t, y_t) > I(a_t, y_t') = a'_{t+1}$. Then, repeating verbatim the argument of part 1 of the proof of Theorem 3 [1], we obtain

$$f(x_t, y_t) = h(a_t, a_{t+1}) \uparrow a_{t+1},\ a_{t+1} \leqslant a^*_{t+1},\ y_t \in B(x_t, p_t), \tag{A.21}$$

$$u(a_t, a_{t+1}) \uparrow a_{t+1},\ a_{t+1} \leqslant a^*_{t+1}. \tag{A.22}$$

Now let $y_t \in A(x_t, p_t)$ be such that $a_{t+1}=I(a_t, y_t) > a^*_{t+1}$. Since $\Sigma \in \Pi$, using (9) we obtain

$$f(x_t, y_t) < f(x_t, x_t). \tag{A.23}$$

Therefore, the function f decreases in a_{t+1} for $a_{t+1} > a^*_{t+1}$. Moreover, it decreases in a_t. Finally, for $y_t' \in A(x_t, p_t)$ we have from the definition of $A(x_t, p_t)$

$$f(x_t, y_t) = f(x_t, y_t'), \quad I(a_t, y_t) = I(a_t, y_t'). \tag{A.24}$$

The last equality holds because $A(x_t, p_t) \subset W(p_t)$. Using (A.24), (A.25) and $x_t = \pi(a_t)$, we have for $a_{t+1} > a^*_{t+1}$

$$f(x_t, y_t) = h(a_t, a_{t+1}) < h(a_t, a^*_{t+1}) = f(x_t, x_t). \tag{A.25}$$

Since the AFM Σ is correct, then by Theorem 1 $V(\Sigma, p_t) \subset B(x_t, p_t) = x_t$. Substituting (A.21) and (A.25) in conditions (10)-(12) of Theorem 1 and using definition (17), we have

$$V(\Sigma, p_t) = \arg \max_{y_t \in \bigcup_{\zeta \in [b, \zeta_t]} A(x_t, a_t, \zeta)} u(a_t, a_{t+1}) = x_t. \tag{A.26}$$

Thus,

$$\arg \max_{a_{t+1} \in A_{t+1}} u(a_t, a_{t+1}) = a^*_{t+1},\ a_{t+1} > a^*_{t+1}. \tag{A.27}$$

Combining (A.21) and (A.25) and also (A.22) and (A.27), we obtain (16). This concludes the proof of necessity.

Sufficiency. Since $a^*_{t+1} \leqslant I(a_t, z_t)$, $z_t \in W(p_t)$ we have by (16) $\arg \max_{a_{t+1} \in A_{t+1}} u(a_t, a_{t+1}) = a^*_{t+1}$. On the other hand using definition (11), (12) of $V(\Sigma, p_t)$ and conditions (14)-(17), we again obtain (A.26). Hence, seeing that $a^*_{t+1} = I((a_t, \pi(a_t))$, we have $V(\Sigma, p_t) = \pi(a_t)$. But $\pi(a_t) = x_t = B(x_t, p_t)$ so that $V(\Sigma, p_t) = B(x_t, p_t)$ and condition (10) of Theorem 1 holds. Thus, the AFM Σ is correct by Theorem 1. Q.E.D.

Proof of Theorem 3. It follows from Theorem 1 that (22) is necessary and sufficient for (8) to hold. Now, using the definitions (21) and (5), we have $\widehat{w}(x_t) = E_{\zeta_t} M_t \widehat{w}(x_t, y_t) = E_{\zeta_t} M_t v(x_t, y_t) = E_{\zeta_t} v(x_t, y_t)$. The second equality holds by (A.13) and the third by (22). Thus, $Q(\Sigma, X) = \operatorname{Arg} \max_{x_t \in X} E_{\zeta_t} v(x_t, z_t)$. Hence $Q(\Sigma, X) \subset B(X)$ if and only if (23) holds. Hence, using the definition of optimal open AFM, we obtain the sought proposition. Q.E.D.

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