ASYMPTOTIC AGGREGATION

G. R. Adilov and V. I. Opoitsev

UDC 62-501.5

We study systems for which aggregation is asymptotically ideal as the system size increases, i.e., the errors of the simplified macro description, in statistical average sense, tend to zero.

1. Introduction

The description of complex technical and economic systems usually relies on aggregated variables. The reasons for this are obvious: the detailed description is unwieldy and incomprehensible, does not fit in computer memory, the large problem cannot be solved in acceptable time, etc. Yet there is another, even weightier, reason. Detailed micro description is usually superfluous, useless. Consider, for instance, a large-scale optimization problem. Even if an exact solution can be obtained, it very seldom means much. The point is that when we attempt to implement the computed solution in practice, we run into various errors, random faults, inconsistencies. As a result, the system actually functions in a suboptimal mode.

The aggregation method is basically studied by two techniques. The first technique looks for systems which admit ideal aggregation without any information losses. Such systems are of course unique, rare, and their study is not generalizable to standard situations. The second technique is more pragmatic. It is concerned with what we may term "forced aggregation methods," which forcibly define certain aggregates and subsequently estimate the losses associated with the aggregated description [1].

In this paper, as in [2], we examine the aggregation issue from an entirely different angle. We study systems in which aggregation is asymptotically ideal as the system size increases. Asymptotically ideal aggregation in this context means that the errors of the simplified macro description, in the statistical average sense, tend to zero as the system size increases.

It should be stressed that the examples considered below are nothing more than models: they are intended to illustrate the general idea and do note pretend to be of any use in practice.

2. Aggregation of the Cobb–Douglas Production Function

Production is often described by the Cobb–Douglas function

$$s(\mathbf{x})=x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}, \tag{1}$$

where

$$\alpha_1+\alpha_2+\dots+\alpha_n=1, \quad \alpha_i\geqslant 0, \quad i=\overline{1,n}. \tag{2}$$

For large systems, it is necessary to estimate the value of $s(\mathbf{x})$ using several aggregates of the form $X^i=\sum_j a_{ij}x_j$. For simplicity, consider the estimation of $s(\mathbf{x})$ using only one aggregate

$$X=\frac{x_1+\dots+x_n}{n}. \tag{3}$$

At a first glance, it seems improbable that the value of a single variable X will produce a reliable estimate of $s(\mathbf{x})$. Yet, for sufficiently large systems, this is actually so.

Moscow. Translated from Avtomatika i Telemekhanika, No. 1, pp. 131-140, January, 1989. Original article submitted November, 19, 1987.

0005-1179/89/5001-0101 $12.50

Anticipating our full analysis and ignoring some details, we can state the corresponding result in the following terms.

For sufficiently large n, we have for almost all vectors x with any desired accuracy

$$s(\mathbf{x})=e^{-c}X=e^{-c}\frac{x_1+x_2+\ldots+x_n}{n}, \tag{4}$$

where C is the Euler constant. Denote by Q the simplex

$$Q=\{x\,|\,x_1+x_2+\ldots+x_n=h,\ \ x_i\geqslant 0,\ i=\overline{1,n}\},$$

whose area is

$$S=\int\limits_{x\in Q}\ldots\int dS=\frac{\sqrt{n}\,h^{n-1}}{(n-1)!}.$$

Using the Dirichlet formula

$$\int\limits_{\substack{x_1+\ldots+x_n\leqslant 1\\ x_1\geqslant 0,\ldots,x_n\geqslant 0}}\ldots\int x_1^{p_1-1}\ldots x_n^{p_n-1}\,dx_1\ldots dx_n=\frac{\Gamma(p_1)\ldots\Gamma(p_n)}{\Gamma(p_1+\ldots+p_n)}$$

we can easily evaluate the surface integral

$$\int\limits_{x\in Q}\ldots\int s(\mathbf{x})\,dS=\frac{\sqrt{n}\,A_n h^n}{n!}, \tag{5}$$

where

$$A_n=\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_n+1).$$

In (5), $s(\mathbf{x})$ is defined by formula (1) subject to (2).

The average value of $s(\mathbf{x})$ on the simplex Q is

$$\bar{s}(\mathbf{x})=\frac{1}{S}\int\limits_{x\in Q}\ldots\int s(\mathbf{x})\,dS=A_n\frac{h}{n}=A_nX. \tag{6}$$

LEMMA 1. Let $n\to\infty$ and

$$\max_j\alpha_j<\frac{M}{n}<1. \tag{7}$$

Then $A_n\to e^{-C}$.

Let us first consider a simple situation when all $\alpha_i = 1/n$. Then

$$\ln A_n=n\ln\Gamma\left(1+\frac{1}{n}\right)=n\left\{\Gamma'(1)\frac{1}{n}+o\left(\frac{1}{n}\right)\right\}.$$

Noting that $\Gamma'(1) = -C$, we obtain $A_n\to e^{-C}$.

In general

$$\ln A_n=\sum_{i=1}^{n}\ln\Gamma(1+\alpha_i)=\Gamma'(1)\sum_{i=1}^{n}\alpha_i+O\left(\sum_{i=1}^{n}\alpha_i^2\right)=$$

$$=\Gamma'(1)+o\left(\sum_{i=1}^{n}\alpha_i\right)$$

and the conclusion is the same. Q.E.D.

Condition (7) ensures that

$$\sum_{i=1}^{n}\alpha_i^2=o\left(\sum_{i=1}^{n}\alpha_i\right), \tag{8}$$

which is precisely what we need. Therefore (7) can be replaced by any other condition which leads to (8).

Let us now check that the average (6) provides a sufficiently accurate estimate of the function $s(\mathbf{x})$ for most vectors $\mathbf{x}\in Q$. To this end, we will show that the relative mean square error of the estimate tends to zero as $n \to \infty$.

First let us determine the "variance"

$$\sigma_n^2 = \frac{1}{S}\int\limits_{\mathbf{x}\in Q} \dots \int \left\{ x_1^{\alpha_1} \dots x_n^{\alpha_n} - \frac{A_n h}{n} \right\}^2 dS =$$

$$= \frac{1}{S}\int\limits_{\mathbf{x}\in Q} \dots \int x_1^{2\alpha_1} \dots x_n^{2\alpha_n}\, dS - \frac{A_n^2 h^2}{n^2} = \left\{ \frac{\bar{A}_n}{n(n+1)} - \frac{A_n^2}{n^2} \right\} h^2,$$

where

$$\bar{A}_n = \Gamma(1+2\alpha_1) \dots \Gamma(1+2\alpha_n).$$

The square of the relative error is given by

$$\frac{\sigma_n^2}{\bar{s}^2(\mathbf{x})} = \frac{n}{n+1}\frac{\bar{A}_n}{A_n^2} - 1.$$

Given (8), we obtain

$$\ln \bar{A}_n = C\sum_i 2\alpha_i + o\left(\sum_i \alpha_i\right) = -2C + o\left(\sum_i \alpha_i\right),$$

$$\ln A_n^2 = 2\ln A_n = -2C + o\left(\sum_i \alpha_i\right).$$

These decompositons, using the previous formulas, easily lead to the following result.

TEOREM 1. Let $n \to \infty$ and let condition (7) (say) hold. Then the average value (6) of the function $s(\mathbf{x})$ gives in the limit the exact value of $s(\mathbf{x})$ for almost all $\mathbf{x}$ in the sense that the relative mean square error $s(\mathbf{x}) - \bar{s}(\mathbf{x})$ tends to zero.

Theorem 1 is of definite interest in itself, because the Cobb–Douglas function is widely used in practice. However, its primary significance is as a reference point, demonstrating the error in the prevailing conviction that increase of problem size invariably means increase of complexity. Here the situation is the exact opposite. The larger the size, the simpler it is to compute the function $s(\mathbf{x})$. We do not need a detailed description of all the resources – it suffices to have only one aggregate.

3. Resource Allocation

Let us now consider the problem of allocation of a single resource

$$\sum_{i=1}^{n} \varphi_i(x_i) \to \max, \quad \sum_{i=1}^{n} x_i = R. \tag{9}$$

For definiteness let $\varphi_i(x_i) = r_i\sqrt{x_i}$. Let us determine the average effect

$$s(\mathbf{x}) = \sum_{i=1}^{n} r_i\sqrt{x_i},$$

when the vectors $\mathbf{x}$ are uniformly distributed on the hyperplane $\sum_i x_i = R$.

Clearly ($\mathbf{x}\geqslant 0$ is assumed throughout),

$$\int\limits_{x_1+\dots+x_n\leqslant R} \dots \int \sqrt{x_1}\, dx_1 \dots dx_n = \int_0^R \left\{ \int\limits_{x_1+\dots+x_n\leqslant R-x_1} \dots \int \sqrt{x_1}\, dx_2 \dots dx_n \right\} dx_1 =$$

$$= \int_0^R \sqrt{x_1}\,\frac{(R-x_1)^{n-1}}{(n-1)!}\, dx_1 = \frac{R^{n+1/2}}{(n-1)!}\int_0^1 z^{1/2}(1-z)^{n-1}\, dz = \frac{R^{n+1/2}}{(n-1)!} B\left(\frac{3}{2}, n\right) = \frac{\Gamma(3/2)}{\Gamma(n+3/2)} R^{n+1/2}.$$

Writing out a similar integral over the domain $x_1 + \ldots + x_n \le \lambda R$, subtracting it from the previous integral, dividing the difference by layer thickness, and letting $\lambda \to 1$, we obtain,

$$\int \ldots \int_{x_1+\ldots+x_n=R} \sqrt{x_1}\, dS = \frac{\Gamma(3/2)\,(n+1/2)}{\Gamma(n+3/2)\sqrt{n}} R^{n-1/2}. \tag{10}$$

Seeing that

$$\Gamma\left(n+\frac{3}{2}\right)=\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right), \quad \Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2},$$

and dividing (10) by the area of the simplex $x_1 + \ldots + x_n = R$, we obtain

$$\frac{1}{S}\int \ldots \int_{x_1+\ldots+x_n=R} \sqrt{x_1}\, dS = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n)}{\Gamma(n+1/2)} R^{1/2}.$$

We now easily obtain the average

$$\bar{s}=\frac{1}{S}\int \ldots \int_{x_1+\ldots+x_n=R} \sum_i r_i\sqrt{x_i}\, dS = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n)}{\Gamma(n+1/2)}\sum_{i=1}^{n} r_i R^{1/2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n)\sqrt{n}}{\Gamma(n+1/2)}\sum_{i=1}^{n} r_i\sqrt{X},$$

where, as before, $X = (x_1 + \ldots + x_n)/n$.

Stirling's expansion of the gamma-function easily gives

$$\Gamma(n)\sqrt{n}/\Gamma(n+1/2)\to 1 \quad \text{as} \quad n\to\infty.$$

Thus, for sufficiently large n, the average effect of resource allocation is given with any accuracy by the formula

$$\bar{s}=\frac{\sqrt{\pi}}{2}\sum_{i=1}^{n} r_i\sqrt{X}. \tag{11}$$

Here, as in the previous section, we can show, under natural assumptions on the coefficients r_i, that the relative mean square error tends to zero as $n \to \infty$. Thus, for large problems, the effect of any allocation **x** is determined with very high probability by the right-hand side of (11).

It is interesting to determine the difference between (11) and the optimum (the maximum attainable effect). The solution of the problem (9) is

$$x_i = \frac{r_i^2}{r_1^2+\ldots+r_n^2} R.$$

For this allocation, the effect is given by

$$s^*=\sqrt{r_1^2+\ldots+r_n^2}\sqrt{R}=\sqrt{r_1^2+\ldots+r_n^2}\sqrt{n}\sqrt{X}.$$

In the simplest case, when all r_i are equal, we have

$$\frac{\bar{s}}{s^*}=\frac{\sqrt{\pi}}{2}\approx 0.89. \tag{12}$$

The relationship (12) holds also in more general cases, e.g., when all r_i are of the same order of magnitude.

From (12) and the above discussion it follows that for large systems an arbitrary allocation is roughly 10% worse than the optimal allocation. The analog of (12) for the previous problem is somewhat worse – around 0.6.

These figures suggest that we should rethink the usefulness of optimization of complex systems, at least the usefulness of detailed optimization. A gain of 10%, of course, can be quite substantial. However, even if the optimal solution is found, it is not simple to implement in a complex system. In practice, we are faced with various unanticipated factors (spoilage or delayed delivery of the resource, an accident or lack of coordination may prevent us from achieving the expected effect, errors in processing and transmission of information, etc.).

Our thesis, of course, should not be viewed as a declaration against optimization. Low effectiveness of optimization is only one of the phenomena that we witness in complex systems. In order to estimate the contribuiton of this phenomenon, we need a specific approach to the analysis of specific problems.

Returning to the resource allocation problem, we have to stress the following point. For small problems, the ratio of the average to optimal effect is different from (12), but it is also fairly close to 1. Nevertheless, in small problems, the situation is qualitatively different. The optimal effect has a flatter peak, and therefore the set of those x on which the effect is below average has a fairly large measure.

4. Optimization by Aggregates

Optimization by aggregates should be applied in systems where detailed optimization is impracticable due to large size, inaccurate modeling, implementation losses, etc.

Let us consider a simple example. We solve the resource allocation problem (9) in which the producers are divided into two groups and the top level first has to decide on the allocation of the resource between these two groups. In other words, the system has to solve the problem

$$\sum_{i=1}^{n} q_i\sqrt{x_i}+\sum_{i=1}^{n} r_i\sqrt{y_i}\to\max,\quad \sum_{i=1}^{n} x_i+\sum_{i=1}^{m} y_i=R, \tag{13}$$

but the management hierarchy is such that the resource is first divided into two parts:

$$X=\sum_{i=1}^{n} x_i,\qquad Y=\sum_{j=1}^{m} y_j,$$

and then each group allocates its appropriation independently.

The top management has to decide between two alternatives: either solve the problem

$$\sqrt{q_1^2+\ldots+q_n^2}\sqrt{X}+\sqrt{r_1^2+\ldots+r_m^2}\sqrt{Y}\to\max,\ X+Y=R \tag{14}$$

or solve the problem

$$\frac{\sqrt{\pi}}{2\sqrt{n}}(q_1+\ldots+q_n)\sqrt{X}+\frac{\sqrt{\pi}}{2\sqrt{m}}(r_1+\ldots+r_m)\sqrt{Y},\quad X+Y=R. \tag{15}$$

From the point of view of optimization of (13), the top management must solve the problem (14). But this is justified only if we assume that the resource will then be allocated optimally within each group. If the groups allocate the resource suboptimally, then the solution of (14) is not necessarily the best. Here the top management should look at the aggregated total effect functions (11) and solve the problem (15) instead of (14).

This example again emphasizes the important fact that aggregated subsystem descriptions are needed not only for simpler computations. In stepwise solution of optimization problems, they are simply indispensable.

5. Selection of Aggregates

Let us return to the problem of aggregation of the function (1). Instead of aggregate (3), we can take another aggregate, linear or nonlinear. For example, take

$$Q=\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}}$$

Let us compute the average of $s(\mathbf{x})$ on the surface $x_1^2 + \ldots + x_n^2 = R^2$ ($x_1 \geq 0$, $x_2 \geq 0$, $\ldots$, $x_n \geq 0$).

Using the Dirichlet formula and making a substitution of variables, we easily evaluate the integral

$$\int\limits_{x_1^2+\ldots+x_n^2=R^2}\ldots\int x_1^{\alpha_1}\ldots x_n^{\alpha_n}\,dx_1\ldots dx_n=$$

$$= \frac{n+1}{2^n} \frac{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}\right)\ldots\Gamma\left(\frac{1}{2}+\frac{\alpha_n}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)} R^n.$$

Dividing this expression by the area of the surface

$$S = \frac{n\pi^{n/2}}{2^n\Gamma\left(\frac{n}{2}+1\right)} R^{n-1},$$

we obtain

$$\frac{1}{S}\int\limits_{x_1^2+\ldots+x_n^2=R^2}\ldots\int x_1^{\alpha_1}\ldots x_n^{\alpha_n}\,dx_1\ldots dx_n = \frac{n+1}{n}\frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}\right)\ldots\Gamma\left(\frac{1}{2}+\frac{\alpha_n}{2}\right)}{\pi^{n/2}\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)}\sqrt{n}\,Q.$$

Now, for $n \to \infty$

$$\Gamma\left(\frac{n}{2}+1\right)\Big/\Gamma\left(\frac{n}{2}+\frac{3}{2}\right) \to 1\Big/\sqrt{\frac{n}{2}+1} \to \sqrt{\frac{2}{n}},$$

$$\ln\left\{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}\right)\ldots\Gamma\left(\frac{1}{2}+\frac{\alpha_n}{2}\right)\Big/\ \pi^{n/2}\right\} \to \frac{1}{2}\Gamma'\left(\frac{1}{2}\right)\Big/\ \Gamma\left(\frac{1}{2}\right) = -\frac{C}{2} - \ln 2,$$

and we obtain the asymptotic bound

$$\bar{s}(\mathbf{x}) \to \frac{e^{-C/2}}{\sqrt{2}}\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}}.$$

Now, as in Sec. 2, we can show that the relative mean square error tends to zero. Therefore, in large systems, we can successfully use the formula

$$s(\mathbf{x}) \approx \frac{e^{-C/2}}{\sqrt{2}}\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}}. \qquad (16)$$

The formula (4) was previously derived for the same purpose. Which of the two formulas is better? The answer depends on additional factors. If in multiple computations the vectors $\mathbf{x}$ are uniformly distributed on the surface $x_1 + x_2 + \ldots + x_n = \text{const}$, then (4) is better; if the vectors $\mathbf{x}$ are uniformly distributed on the sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = \text{const}$, then (16) is preferable.

The following example illustrates the aggregate selection problem. Assume that the vectors $\mathbf{x}$ are uniformly distributed on the euclidean sphere, but we make a "hardheaded decision" and estimate $s(\mathbf{x})$ as a function of the aggregate X, for example

$$s(\mathbf{x}) = \nu\frac{x_1+\ldots+x_n}{n}. \qquad (17)$$

In order to determine the asymptotically best constant ν, we need to evaluate the integral

$$\frac{1}{S}\int\limits_{x_1^2+\ldots+x_n^2=R^2}\ldots\int\frac{ns(\mathbf{x})}{x_1+\ldots+x_n}\,dx_1\ldots dx_n,$$

(this is quite easy) and pass to the limit as $n \to \infty$ (using $R = \sqrt{n}Q$).

As a result, we obtain a coefficient ν which is not e^{-C}, although it is close to e^{-C}. Which estimate should we use, (4) or (17)? Under these conditions, (17) is preferable, although its relative mean square error after averaging over the simplex $x_1 + x_2 + \ldots + x_n = \text{const}$ no longer tends to zero.

The point is that the ratio $s(\mathbf{x})/X$ is strictly constant on rays. The choice of a particular surface with a uniform measure for averaging in general induces a nonuniform measure on rays and thus affects the computed averaged characteristics.

6. Numerical Experiments

The results of Secs. 2, 3, 5 show that, for sufficiently large systems, the micro description may be replaced with fair accuracy by a macro description. Since the results are only valid in the limit and we have not estimated the rate of convergence, the expressions "sufficiently large" and "fair accuracy" are undefined. In order to determine the particular system size when we can pass to a macro description and to estimate its accuracy, we have conducted a number of numerical experiments.

1. The vectors $\mathbf{x}=(x_1,\ldots,x_n)$ are generated by a random number generator. In order to obtain a uniform distribution on the simplex $x_1 + x_2 + \ldots + x_n = \text{const}$, $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_n \geq 0$, each x_i is defined as an independent realization of the random variable $-\frac{1}{\lambda}\ln\xi_i$, where ξ_i is a [0, 1] uniform variate. α_i are generated as realizations of the random variable $\xi_i M + 1$, which after normalization ensures that (7) is satisfied.

Below we report the results of experiments with $\lambda = 10$, $M = 5$.

The function $s(\mathbf{x})=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ was evaluated nearly a thousand times for various (normalized) vectors x. For n = 10, 50, 100, 200, the ratio $\bar{s}(\mathbf{x})/X$, respectively had the values ($e^{-c} = 0.5616$).

$$\bar{s}(\mathbf{x})/X=e^{-c}+0.0517;\ e^{-c}+0.0091;\ e^{-c}+0.0047;\ e^{-c}+0.0017.$$

The relative variance respectively was

$$\sigma_s/\bar{s}(\mathbf{x})=0.2731;\ 0.1361;\ 0.1006;\ 0.0703.$$

2. The next cycle of experiments performed the same computations for the aggregate $X = \sqrt{(x_1^2 + \ldots + x_n^2)/n}$. The uniform distribution on the sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = \text{const}$, $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_n \geq 0$, was generated by taking each x_i as an independent nonnegative realization of a $(0, \sigma)$ normal variate on $[-\infty, +\infty]$.

Below we report the experimental results for $\sigma = 30$.

As in the previous cycle of experiments, the function $s(\mathbf{x})=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ was evaluated around a thousand times on a sphere. The corresponding values for n = 10, 50, 100, 200 were the following ($e^{-c/2}/\sqrt{2} = 0.5302$):

$$\frac{\bar{s}(\mathbf{x})}{X}=\frac{e^{-c/2}}{\sqrt{2}}+0.0485;\quad \frac{e^{-c/2}}{\sqrt{2}}+0.0157;\quad \frac{e^{-c/2}}{\sqrt{2}}+0.0079;$$

$$\frac{e^{-c/2}}{\sqrt{2}}+0.0026;\quad \frac{\sigma_s}{\bar{s}(\mathbf{x})}=0.2378;\quad 0.1123;\quad 0.0821;\quad 0.0391.$$

3. The third cycle of experiments focused on the resource allocation problem. The vectors x were generated by the same random number generator as in the first cycle, with r_i defined as α_i. The function $s(\mathbf{x})=\sum_{i=1}^{n} r_i\sqrt{x_i}$ was evaluated around a thousand times on a simplex. The corresponding results for n = 10, 50, 100, 200 were $\left(\sqrt{\pi}\sum_i r_i/2=0.8861\right)$:

$$\frac{\bar{s}(\mathbf{x})}{\sqrt{X}}=\frac{\sqrt{\pi}\sum_i r_i}{2}+0.0093;\quad \frac{\sqrt{\pi}\sum_i r_i}{2}+0.0023;$$

$$\frac{\sqrt{\pi}\sum_i r_i}{2}+0.0015;\quad \frac{\sqrt{\pi}\sum_i r_i}{2}+0.0005;$$

$$\frac{\sigma_s}{\bar{s}(\mathbf{x})}=0.0859;\quad 0.0343;\quad 0.0256;\quad 0.0191.$$

Thus, experimental results make it possible to estimate, subject to specified accuracy, the system size when the macro description becomes acceptable.

LITERATURE CITED

1. A. A. Pervozvanskii and V. G. Gaitsgori, Decomposition, Aggregation, and Approximate Optimization [in Russian], Nauka, Moscow (1979).
2. G. R. Adilov and V. I. Opoitsev, "Aggregation of linear systems," Avtomat. Telemekh., No. 8, 64-71 (1986).

M-MACHINE PROBLEM WITH REAL-TIME PROCESSING

V. I. Levin

UDC 65.012.122

The Bellman–Johnson problem for time optical sequencing of an jobs on M machines given the same processing sequence of the jobs on each machine is generalized to the case of real-time arrival of jobs, i.e., the next job to be processes is not necessarily waiting at the system input when the current job is being processed. We show that time optimization of a real-time system reduces to the solution of a similar problem for a batch-processing system.

1. Introduction

The problem of the optimal sequencing of a batch of jobs on 2 machines (the Bellman–Johnson problem) was solved in [1]. This was the beginning of scheduling theory. Modern scheduling theory [2, 3] usually considers situations when the entire batch of jobs to be processed is already present in the system before processing begins. This is called batch processing. In practice, this situation is observed when a special buffer bin provided at the input to the system of machines is replenished by a sufficiently productive technological source, so that a batch of sufficient size is always available for processing. However, if batch processing conditions are not observed (e.g., there is no buffer bin), we are faced with a qualitatively new situation, when only a partial set of jobs, or possibly no jobs at all, are present at the system input before processing begins. Thus, the jobs are delivered to teh machines directly as they arrive in the system, so that instead of batch processing we have individual or unit processing, which we call real-time processing.

In this paper, we propose an approach to optimzing the processing sequence of jobs arriving in real time. This is a generalization of the previously proposed approach to a solution of a similar problem for a batch processing system, which relies on special tools (infinite-valued logic and logical determinants) to obtain a comprehensible description of multidimensional problems [4, 5].

2. Statement of the Problem

Consider a system of m serially connected machines operating in real time. We will restrict the analysis to a time interval during which a bounded set of jobs $\{1, 2, \ldots, n\}$ arrive in the system for processing. These jobs arrive in a fixed order $P_n = (j_1, j_2, \ldots, j_n)$, as in batch processing [1, 3, 5], but at unknown real-time moments $t_{j1}, t_{j2}, \ldots, t_{jn}$, $t_{j1} \leq t_{j2} \ldots \leq t_{jn}$, which form a vector $t = (t_{j1}, \ldots, t_{jn})$ of arrival times. Here j_k is the index of the k-th arriving job, t_{kj} is its arrival time ($k = \overline{1, n}$). The jobs are processed as in batch mode [1, 3, 5]. Specifically, a certain processing sequence is chosen $P = (i_1, i_2, \ldots, i_n)$, where i_k is the index of the k-th job in the processing sequence ($k = \overline{1, n}$), and in general the processing sequence P is not the same as the arrival sequence P_n. All the jobs are processed in a fixed sequence P, which is the same for all machines. Each job from P is processed sequentially first on machine 1, then on machine 2, ..., and finally on machine

---

Perm'. Translated from Avtomatika i Telemekhanika, No. 1, pp. 141-154, January, 1989. Original article submitted June 23, 1987.

0005-1179/89/5001-0108$ $12.50