COMPETITION MECHANISMS FOR ALLOCATION OF SCARCE RESOURCES

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We investigate the efficiency of competition mechanisms for the allocation of scarce resources. We show that for certain classes of effect functions (production functions) competition mechanisms ensure optimal allocation of resources betwen the winners. A connection is established between competition mechanisms and open management mechanisms.

1. Introduction

Competition mechanisms are attracting ever increasing attention in management theory and practice. The specific feature of competition mechanisms is that the players (the elements of the system) participate in a competition organized in the resource-allocation part of the planning procedure. The competition winners are the elements that achieve the highest efficiency measures of resource utilization in the plans submitted to the headquarters. The competition winners are awarded a certain priority in allocation of resources. The attempt to win the competition encourages the elements to submit efficient plans.

Competition mechanisms are successfully implemented in the management of Bulgarian national economy. Examples of competition mechanisms include the national competition of proposals for the development of small and medium enterprises; the license-buying competition of the State Committee of Science and Scientific-Technical Progress; the competition for the allocation of capital investment and foreign exchange budgets for the development of small and medium enterprises for the production of new materials; a national competition of credit allocation proposals for capital investments for implementation of the plans submitted by business enterprises.

The competition principle of planning has been adopted to a certain extent in the procedures for evaluation of the production programs of the enterprises controlled by the USSR Ministry of Instrument Building. This procedure analyzes the plans submitted by the enterprises (industrial associations) in order to determine the number of enterprises that will receive the minium (base) appropriation of centralized capital investments. The other enterprises (competition winners) receive larger capital investment appropriations in accordance with the performance indicators of their plans.

On the theoretical level, competition mechanisms are classified as so-called multichannel organizational mechanisms of the theory of active systems [1]. In this paper, we investigate the competition mechanism of allocation of scarce resources in an active system consisting of a center, which controls resource allocation, and active elements, which represent the resource users. The competition winners receive the requested quantity in full, while the remaining elements receive only the minimum quantity set by the center. We prove the existence of a Nash equilibrium in the corresponding game and show that, under certain conditions, this mechanism produces an optimal allocation of resources between the competition winners.

2. Description of the Model. The Competition Mechanism

Consider an active system comprising n elements and a center that allocates a scarce resource. Let R be the quantity of the resource available at the center, x_i the quantity of the resource received by the i-th element, u_i the effect produced by the use of the resource by element i. In practice, the effect is usually identified with increase of output or with some integrated estimate of the effect by a number of criteria. We take $u_i=\varphi_i(x_i)$, where

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$\varphi_i(x_i)$ is the effect function or the production function. We assume that the function $\varphi_i(x_i)$ is strictly concave, differentiable for $x_i > 0$, nondecreasing, and defined for $x_i \geq 0$; also $\varphi_i(0) = 0$.* The problem of the center is to determine the resource allocation $x = \{x_i, i = 1-n\}$ so as to maximize the overall effect

$$\Phi(x)=\sum_{i=1}^{n}\varphi_i(x_i) \tag{1}$$

subject to the constraint

$$\sum_{i=1}^{n} x_i \leqslant R. \tag{2}$$

The problem (1), (2) is difficult to solve because, in general, the center does not have sufficiently accurate information about the functions $\varphi_i(x_i)$. The information about the local behavior of the functions $\varphi_i(x_i)$ is communicated to the center by the elements in the form of so-called retrogressive or divisional plans (s_i, w_i), where s_i is the requested quantity of the resource and w_i is the estimate of the (expected) achieved effect. The active behavior of the elements is manifested in their tendency to exaggerate upward the requested quantities s_i. If the elements are reouired to pay for the resources, then the optimal resource allocation in active system theory is achieved by so-called open-management mechanisms [2]. If the resources are free, but sufficiently severe penalities (sanctions) are imposed for nonfulfillment of the planned (expected) effect, then the optimal allocation for a wide class of production functions is achieved by progressive optimal allocation mechanisms [2], auction schemes [3], or the reverse priority principle [4]. So far, however, no optimal allocation mechanisms have been proposed for the case when the resources are free and there are no sufficiently severe penalities for nonfulfillment of the planned effect.

Let us define the competition mechanism. We denote by $q_i = u_i/x_i$ the efficiency of resource utilization by the i-th element, by $\xi_i = w_i/s_i$ the efficiency estimate as reported by the i-th element. Arrange ξ_i in increasing order, i.e., $\xi_{i_1} \geq \xi_{i_2} \geq \ldots \geq \xi_{i_n}$. Initially, we ignore the constraint (2) on resource availablility at the center.

Definition. A competition mechanism is a resource allocation mechanism in which the planning procedure includes a stage that identifies a set Q of elements called the competition winners. This set consists of m elements with the highest efficiency estimates, i.e., $Q = \{i_k : k \leq m\}$, where $m < n$. The resource allocation procedure given the set of winners has the form

$$x_i=\begin{cases} s_i, & \text{if} \quad i\in Q, \\ c, & \text{if} \quad i\notin Q, \quad i\in I\setminus Q, \end{cases} \tag{3}$$

where c is the minimum awarded level of the resource $(c > 0)$, $I = \{1, 2, \ldots, n\}$. The objective functions of the elements are taken in the form

$$f_i(u_i, w_i)=u_i-\psi_i(w_i-u_i), \tag{4}$$

where

$$\psi_i(w_i-u_i)=\begin{cases} \alpha(w_i-u_i), & \text{if} \quad w_i-u_i\geqslant 0, \\ 0, & \text{if} \quad w_i-u_i<0 \end{cases}$$

(by definition, for $i\notin Q$, we set $w_i = u_i = \varphi_i(c)$, $\alpha > 0$). Here the function $\psi_i(w_i - u_i)$ is the penalty for nonfulfillment of the expected effect w_i. We assume that each element selects its strategies (the reported s_i and w_i) so as to maximize its objective function (4).

Thus, the choice of the information that the elements report under a competition mechanism may be treated as an n-person game. Note that since $\xi_i = w_i/s_i$, then any pair (s_i, w_i), (s_i, ξ_i) or (w_i, ξ_i) may be used as the reported estimates. For the ease of game-theoretical analysis, we use the pair (s_i, ξ_i) as the reported estimate. For a given ξ_i, the estimate s_i is determined so as to maximize (4), where $u_i = \varphi_i(x_i)$, $w_i = \xi_i s_i$, because the estimate s_i does not affect the selection of competition winners.

*The results remain valid also for twice piecewise-differentiable functions $\varphi_i(x_i)$.

3. Solution of the Game Under the Competition Mechanism and Its Optimality

Let us define the function

$$h_i(\xi_i)=\max_{s_i}[\varphi_i(s_i)-\psi_i(\xi_i s_i-\varphi_i(s_i))]$$

and investigate its properties.

The maximum point of the function $f_i=\varphi_i(s_i)-\psi_i(\xi_i s_i-\varphi_i(s_i))$ over s_i will be denoted by x_i. In what follows, we distinguish between weak and strong penalties, as defined by α. α defines a weak penalty for the i-th element for a given ξ_i if the function $f_i(\varphi_i(s_i),\xi_i s_i)$ attains its maximum over s_i at the point x_i determined by the conditions

$$\left[\varphi_i'(x_i)-\frac{\alpha}{1+\alpha}\xi_i\right]x_i=0 \quad \text{and} \quad \xi_i x_i\geqslant\varphi_i(x_i). \tag{5}$$

α defines a strong penalty for the i-th element for a given ξ_i if the function $f_i(\varphi_i(s_i),\xi_i s_i)$ attains its maximum over s_i at the point x_i determined by the condition

$$\varphi_i(x_i)=\xi_i x_i. \tag{6}$$

The expressions (5) and (6) describe necessary and sufficient conditions of extremum of f_i over s_i in virtue of strict concavity of the function f_i.

Let $\gamma_i(\xi_i)$ be the value of x_i satisfying (5) and $\beta_i(\xi_i)$ the value of x_i satisfying (6).

<u>LEMMA</u>. a) The function $h_i(\xi_i)$ is continuous and decreasing in ξ_i;

b) $\sup_{0<\xi_i<\infty} h_i(\xi_i)\geqslant\varphi_i(c)$;

c) $\lim_{\xi_i\to\infty} h_i(\xi_i)=0$.

The proof is given in the Appendix.

It follows from the lemma that the equation $h_i(\xi_i) = c$ always has a unique solution. Denote the root of this equation by $v_i = v_i(c)$.

First consider the case when

$$\sum_{i\in Q} x_i+(n-m)c<R, \tag{7}$$

i.e., we assume that the total resource availability is unlimited.

The functioning of the system under the competition mechanism will be considered as an n-person game in which the strategies of the players (the active elements) are the reported estimates $\{\xi_i\}$, and the payoff functions are $h_i(\xi_i)$ when $i\in Q$ and $\Delta_i=\varphi_i(c)$ when $i\notin Q$. The solution of the game $\xi^* = \{\xi_i^*\}$ is identified with Nash equilibrium.

Let the elements be ordered by decreasing v_i, i.e.,

$$v_{i_1}\geqslant v_{i_2}\geqslant\ldots\geqslant v_{i_n}.$$

<u>THEOREM 1</u>. The solution of the game (Nash equilibrium) under the competition mechanisms exists, and for $j\in Q$, $\xi_j^* = v_{i_{m+1}}$.

The proof is given in the Appendix.

The simplest competition procedure producing an equilibrium in two iterations is the following.

<u>Step 1</u>. The elements report $\xi_i^0 = v_i$, $i = 1, 2, \ldots, n$. The center declares the winners in the current step and orders the elements by decreasing v_i.

<u>Step 2</u>. The winners $j\in Q$ report $\xi_j^0 = v_{i_{m+1}}$, the remaining elements $j\notin Q$ do not revise their estimates $\xi_j^1 = \xi_j^0 = v_j$. An equilibrium has been obtained.

From Theorem 1 it follows that in the solution of the game under the competition mechanism we have ξ_i = const for all $i\in Q$, i.e., in the solution of the game ξ is the same for all the winners.

Let us now consider the construction of noncompetition planning mechanisms which are not less efficient than the competition mechanism in this case.

Assume that the effect function $\varphi_i(z_i, r_i)$ depends on the parameters r_i. The center does not know the value r_i, and it is only known that $r_i \in \Omega_i$, where Ω_i is the set of possible values of the parameter r_i, $i = 1, 2, \ldots, n$. The elements, on the other hand, are assumed to know the exact value of the parameters r_i.

We assume that the elements report the parameter estimates r_i to the center and the center uses this information to allocate the resources. Given the estimates $\sigma = \{\sigma_i\}$ the center calculates the estimates $v_i = v_i(\sigma)$ by solving the equations $h_i(v_i, \sigma_i) = \varphi_i(c, \sigma_i)$, where

$$h_i(v_i, \sigma_i) = \max_{s_i} [\varphi_i(s_i, \sigma_i) - \psi_i(v_i s_i - \varphi_i(s_i, \sigma_i))].$$

Arrange $v_i(\sigma_i)$ in decreasing order, i.e.,

$$v_{i_1}(\sigma_{i_1}) \geqslant v_{i_2}(\sigma_{i_2}) \geqslant \ldots \geqslant v_{i_n}(\sigma_{i_n}),$$

and let

$$\xi_i^* = v_{i_{m+1}}(\sigma_{i_{m+1}}).$$

Consider the coordinated planning problem

$$\sum_{i=1}^{n} \varphi_i(z_i, \sigma_i) \to \max \tag{8}$$

subject to

$$\begin{aligned} &i \in Q : x_i = \arg\max f_i(\varphi_i(z, \sigma_i), \xi^*), \\ &i \notin Q : x_i = c, \end{aligned} \tag{9}$$

where $f_i(\varphi_i(z, \sigma_i), \xi^*) = \varphi_i(z, \sigma_i) - \psi_i(\xi^* z - \varphi_i(z, \sigma_i))$.

In the "coordinated" case, the parameter ξ^* is independent of σ_i, and therefore the procedure described above is in fact an open-management procedure. We know [5] that for open-management procedures truthful reporting of information $\sigma_i = r_i$ is the dominant strategy for all the elements.

Note that for the open-management procedure $\sigma_i = r_i$, $i = 1, 2, \ldots, n$, and therefore the allocation of resources under the competition mechanism coincides with the solution of the problem (8), (9) defining the open-management procedure. We thus have the following theorem.

THEOREM 2. For any competition mechanism of resource allocation (3) in the case (7) there exists an open-management procedure (8), (9) which is not less efficient.

Remark 1. This connection between competition mechanisms and open-management procedures makes it possible to organize a one-step competition procedure. The elements report the estimates $\sigma = \{\sigma_i\}$, the center calculates $v_i\{\sigma_i\}$, identifies the set of winners Q, and sets the corresponding plans.

Let us now consider the question of optimality of competition mechanisms of resource allocation and thus of the corresponding open-management procedures.

Let the total quantity of the resource allocated to winners under the competition mechanism be

$$R' = \sum_{i \in Q} x_i = \sum_{i \in Q} s_i$$

and consider the following resource allocation problem:

$$\begin{gathered} \sum_{i \in Q} \varphi_i(x_i) \to \max_{\{x_i\}}, \\ \sum_{i \in Q} x_i = R'. \end{gathered} \tag{10}$$

As we have noted above, by Theorem 1 we have for all the winners in equilibrium

$$\xi_j^* = v_{i_{m+1}} = \text{const}, \quad j \in Q.$$

Assume that the equilibrium is characterized by weak penalties for all the winners, i.e., for all $j \in Q$, the requests are selected from $\varphi_j'(s_j) = \frac{\alpha}{1+\alpha} v_{i_{m+1}}$. In case of weak penalties for nonfulfillment of the planned effect, we have the following theorem.

THEOREM 3. The set of requests of the winners under the competition mechanism in Nash equilibrium is an optimal solution of the problem (10).

The theorem follows directly from the optimality condition $\varphi_i'(s_i) = \text{const}$ when allocating a total quantity of resources R'. This theorem may be employed to find an approximate solution of the optimal allocation problem (1), (2).

If the ratio $\sum_{i \in I \setminus Q} \varphi_i(x_i) \Big/ \sum_{i \in I} \varphi_i(x_i)$ characterizing the relative effect of the losers in the competition is small, then the approximate solution $\tilde{x} = \{\tilde{x}_i\}$ is given by

$$\tilde{x}_i = \begin{cases} \gamma_i(v_{i_{m+1}}(\tilde{c})), & \text{if} \quad i \in Q, \\ \tilde{c} \quad , & \text{if} \quad i \in I \setminus Q, \end{cases} \tag{11}$$

where

$$\sum_{i \in Q} \gamma_i(v_{i_{m+1}}(\tilde{c})) + (n-m)\tilde{c} = R. \tag{12}$$

Thus, calculation of an approximate solution reduces to finding $\tilde{c}$ such that (12) holds. $\tilde{c}$ can be found by various iterative schemes, in which the estimate c of the parameter $\tilde{c}$ is chosen according to the sign of $\sum = \sum_{i \in Q} \gamma_i(v_{i_{m+1}}(c)) + (n-m)c - R$ (if $\Sigma < 0$, then the center increases c, if $\Sigma > 0$, then the center reduces c, if $\Sigma = 0$, then $\tilde{c} = c$ and a solution has been found).

In these schemes, unlike the previously considered schemes, $\tilde{c}$ is chosen depending on the strategies of the elements. Therefore, the resulting equilibrium may differ from that established in Theorem 1. We will now give some considerations which can be used to estimate the corresponding equilibrium point ξ_i^*.

In order to find the strategies ξ_i^* for $i \in Q$, note that $\gamma_i(v_{i_{m+1}})$ is a decreasing function of $v_{i_{m+1}}$. Thus, the higher $v_{i_{m+1}}$, the higher is c. Since the elements $i \in I/Q$ prefer higher $\tilde{c}$, then at least the element i_{m+1} will report the estimate $\xi_{i_{m+1}}$ such that $v_{i_{m+1}}$ is as high as possible. Since $v_{i_{m+1}} \le v_{i_m}$, then in the solution of the game $\xi^*_{i_{m+1}} = v_m$. The remaining elements ($i \in I/Q$), may report arbitrary estimates $\xi_i^* \le v_{i_m}$.

Thus, if the "weight" of the winners among all the elements is sufficiently high, then the proposed mechanism produces a nearly optimal allocation of resources. To this end, the number of winners should be made as large as possible, i.e., m = n − 1.

Let us consider a particular example in order to demonstrate the use of the competition mechanism and the proximity of the resulting allocation to the optimum.

Example 1. Let $\varphi_i(x_i, r_i) = \sqrt{r_i x_i}$, $i = 1, 2, \ldots, n$, and the number of winners is n − 1. Then the condition (5) takes the form $\frac{\sqrt{r_i}}{2\sqrt{x_i}} = \frac{\alpha}{1+\alpha}\xi$, whence

$$x_i = \left(\frac{1+\alpha}{2\alpha\xi}\right)^2 r_i \quad \text{and} \quad h_i(\xi) = \frac{(1+\alpha)^2}{4\alpha\xi} r_i.$$

Determine $\xi^* = v_n$ from the condition (5), which takes the form

$$\frac{(1+\alpha)^2}{4\alpha\xi} r_{n-1} = \sqrt{r_{n-1} c}.$$

Hence

$$\xi^* = \frac{(1+\alpha)^2}{4\alpha}\frac{\sqrt{r_{n-1}}}{\sqrt{c}},$$

Substituting ξ^*, in the expression for x_i, we obtain the solution

$$x_i^* = \begin{cases} \dfrac{4\tilde{c}}{(1+\alpha)^2}\dfrac{r_i}{r_{n-1}}, & \text{if} \quad i=1,2,\ldots,n-1, \\ \tilde{c}, & \text{if} \quad i=n, \end{cases}$$

$$\tilde{c} = \frac{Rr_{n-1}}{\dfrac{4}{(1+\alpha)^2}H - \dfrac{4}{(1+\alpha)^2}r_n + r_{n-1}}, \quad \text{where} \quad H = \sum_{i=1}^{n} r_i.$$

Substituting $\tilde{c}$ in the expression for x_i^*, we obtain

$$x_i^* = \frac{Rr_i}{H - r_n + r_{n-1}\dfrac{(1+\alpha)^2}{4}} \quad \text{for} \quad i \neq n.$$

Let us estimate the relative error

$$\delta = \frac{\Phi(x^{\text{opt}}) - \Phi(x^*)}{\Phi(x^{\text{opt}})}.$$

We easily see that

$$\Phi(x^{\text{opt}}) = \sqrt{RH}, \quad \Phi(x^*) = \left(\sqrt{RH} - \frac{r_n\sqrt{R}}{\sqrt{H}} + \frac{1+\alpha}{2}\sqrt{\frac{R}{H}}\sqrt{r_n r_{n-1}}\right)B,$$

where

$$B = 1\Big/\sqrt{1 - \frac{r_n}{H} + \frac{r_{n-1}}{H}\frac{(1+\alpha)^2}{4}}.$$

Hence

$$\delta = 1 - \left(1 - \frac{r_n}{H} + \frac{(1+\alpha)}{2}\frac{\sqrt{r_n r_{n-1}}}{H}\right)B.$$

Assume that r_{n-1}/H and r_n/H are sufficiently small. Then, expanding in Taylor series, we obtain

$$\delta \approx \frac{1}{2H}\left[\sqrt{r_n} - \frac{(1+\alpha)}{2}\sqrt{r_{n-1}}\right]^2 \leqslant \frac{r_n + \left(\dfrac{1+\alpha}{2}\right)^2 r_{n-1}}{2H}.$$

Thus, for $0 < \alpha < 1$, the relative error is $\delta \leq r_{n-1}/2H$.

The competition mechanism of resource allocation may be implemented as an open-management procedure with bottom-up flow of information from the elements to the center. Assume that the elements report the estimates σ_i of the parameters r_i. The center calculates the parameter $\xi = \dfrac{(1+\alpha)^2\sqrt{\sigma_n}}{4\alpha\sqrt{c}}$ for all $i \in Q = (1, 2, \ldots, n-1)$ and determines the planning procedure from the formulas (8), (9), where

$$c = \frac{R\sigma_n}{\dfrac{4}{(1+\alpha)^2}\sum_{i=1}^{n-1}\sigma_i + \sigma_n}.$$

For sufficiently large n, c depends on the estimate σ_i for $i \in Q$. Therefore, if we assume that the i-th element ignores the effect of its estimate σ_i on the parameter c, then the conditions (8) determine an open-management procedure. In this case, the elements $i \in Q$ report truthful information $\sigma_i = r_i$. The element n exaggerates the estimate σ_n upward to $\sigma_n(c, r_{n-1})$, satisfying the equation $v_n(c, \sigma_n) = v_{n-1}(c, \sigma_{n-1})$. It is easy to see that in our example $\sigma_n = r_{n-1}$. We thus see that the resource allocation produced by the open-management procedure coincides with the allocation by the competition mechanism. In the open-management procedure, the allocation is computed in one step.

Let us now consider the case of strong penalties for nonfulfillment of the planned effect. In this case, the value of the parameter α is so high that the maxima of the functions f_i over s_i for equilibrium ξ are attained at the points $x_i = \beta_i(\xi)$, i.e., are given by the conditions (6).

We will show that the solution of the game under the competition mechanism in the strong penalty case also has a certain optimality property. Let the functions $\varphi_i(x_i)$ be concave and continuously differentiable for all i = 1, 2, ..., n and let at least two elements have different functions, i.e., $\forall x \exists i, j : \varphi_i(x) \neq \varphi_j(x)$.

THEOREM 4. For $\{x_i | i \in Q\}$ to be an optimal solution of the problem (10) for any $v_{i_{m+1}}$, it is necessary and sufficient that all the functions $\varphi_i(x)$, $i \in Q$ belong to the class of functions N_θ defined by the differential equation

$$d\varphi_i/dx = \theta[\xi(\varphi_i(x), x)], \tag{13}$$

where $\theta(\cdot)$ is an arbitrary one-one function of a single variable.

The proof is given in the Appendix.

Remark 2. In case of piecewise-continuous differentiability φ_i. Theorem 4 remains true, and it is only necessary to replace the class of functions N_θ with the class of functions defined by the inequalities

$$d\varphi_i^-/dx_i \geqslant \theta[\xi(\varphi_i, x_i)] \geqslant d\varphi_i^+/dx_i.$$

Remark 3. If the set N_θ consists of a single function, i.e., $\varphi_i(x) = \varphi(x)$, i = 1, 2, ..., n, then clearly in the solution of the game $x_i^\theta = x^*$ for all $i \in Q$, which is an optimal solution of the problem (10) for all concave functions $\varphi(x)$.

Theorem 4 constructs optimal competition mechanisms for the strong penalty case by the same scheme that we have described above for weak penalties.

A similar optimality theorem for the problem of resource allocation by the auction mechanism was proved by Burkov and Yusupov [6].

Example 2. Let $u_i = \varphi_i(x_i) = r_i x_i^a$, $a < 1$, $i = 1, 2, \ldots, n$. We have $d\varphi_i/dx_i = a r_i x_i^{a-1} = a u_i/x_i$. If we take $\xi = u_i/x_i$, then the competition mechanism produces an optimal allocation of resources between the winners.

Example 3. Let $\xi_i = u_i/x_i$, i = 1, 2, ..., n. Consider the differential equation

$$\frac{d\varphi}{dx} = a\frac{\varphi}{x^2}.$$

Its solution is $\varphi = re^{-a/x}$. Note that

$$\frac{d\varphi}{dx} = \frac{ar}{x^2}e^{-a/x}$$

is a decreasing function only for $x > a/2$. Therefore the competition mechanism with efficiency measure $\xi = u/x^2$ produces an optimal allocation of resources between the winners by the criterion

$$\Phi = \sum_{i \in Q} r_i e^{-a/x_i}$$

for $x_i > a/2$ for all $i \in Q$.

4. Conclusion

Let us summarize our results. We have described a model of a competition mechanism of resource allocation; we have shown that the competition mechanism leads to a game the solution of which is a Nash equilibrium; we have established a connection of the competition mechanism with the open-management principle (for a given competition mechanisms, an open-management procedure can be constructed which is not less efficient than the competition mechanism); we have shown that with weak penalties the competition mechanism (for a sufficiently large number of elements) gives a nearly optimal allocation, whereas with strong penalties we have derived conditions that ensure closeness of the solution of the competition game to the optimum.

Further development and generalization of our results is possible, say, by examining penalty functions $\psi_i(\cdot)$ of a more general form and also by investigating systems with mixed penalties, when the penalties are weak for some elements and strong for the others. The problem can be generalized also by considering allocation of a multidimensional resource (composite deliveries, interchangeable resources). In this case, the existence theorem of Nash equilibrium for the competition mechanism remains true, but the question of optimality requires further analysis.

APPENDIX

Proof of the Lemma.

a) Consider the function $h(\xi)=\alpha\max_x\left[\frac{1+\alpha}{\alpha}\varphi(x)-\xi x\right]$, where the function $\varphi(x)$ is strictly concave and differentiable for $x > 0$, $\varphi(0) = 0$. Let the function $h(\xi)$ be defined at the point $\xi = \xi^0$. We will show that $h(\xi)$ is continuous at this point.

By differentiability and strong concavity of the function $\frac{1+\alpha}{\alpha}\varphi(x)-\xi^0 x$, it attains a maximum over x at the unique point x^0 defined by the condition $\varphi'(x^0)=\frac{\alpha}{1+\alpha}\xi^0$. By strict concavity of the function $\frac{1+\alpha}{\alpha}\varphi(x)-\xi(x)$ in x, it follows that $\frac{1+\alpha}{\alpha}\varphi(x)-\xi$ is decreasing in x.

Since for a fixed ξ, $\frac{1+\alpha}{\alpha}\varphi(x)-\xi x$ is a function of a single variable x, its derivative has no discontinuities of the first kind, and since this derivative is bounded and monotone on $[\varepsilon', \infty]$, where $\varepsilon' > 0$, we obtain that $\frac{1+\alpha}{\alpha}\varphi'(x)-\xi$ has no discontinuities of the second kind. Thus, $\frac{1+\alpha}{\alpha}\varphi'(x)-\xi$ is a continuous decreasing function.

It follows that the equation $\frac{1+\alpha}{\alpha}\varphi'(x)-\xi^0-\delta=0$ for sufficiently small δ (in absolute value) has a unique solution x_δ. Here $|x_\delta - x^0| < \nu_\delta$, where $\lim_{\delta\to 0}\nu_j=0$. Indeed, if $\lim_{\delta\to 0}\nu_\delta=\text{const}>0$, then we obtain that the function $\frac{1+\alpha}{\alpha}\varphi(x)-\xi^0 x$ attains its maximum at least at two different points, which contradicts strict concavity of the function $\frac{1+\alpha}{\alpha}\varphi(x)-\xi^0 x$.

Let $\xi = \xi^0 + \delta$. We will show that $\forall\varepsilon > 0$ $\exists\delta_\varepsilon > 0$, such that $\forall\delta < \delta_\varepsilon$ if $|\xi - \xi^0| < \delta$, then $|h(\xi) - h(\xi^0)| < \varepsilon$, i.e., $h(\xi)$ is continuous. Consider

$$|h(\xi)-h(\xi^0)|=\alpha\left|\max_x\left(\frac{1+\alpha}{\alpha}\varphi(x)-\xi^0 x-\delta x\right)-\max_x\left(\frac{1+\alpha}{\alpha}\varphi(x)-\xi^0(x)\right)\right|=$$

$$=\alpha\left|\frac{1+\alpha}{\alpha}\varphi(x_\delta)-\xi^0 x_\delta-\delta x_\delta-\frac{1+\alpha}{\alpha}\varphi(x^0)-\xi^0 x^0\right|\leqslant(1+\alpha)|\varphi(x^0)-\varphi(x_\delta)|+\alpha\xi^0|x_\delta-x^0|+$$

$$+\alpha\delta x_\delta\leqslant[(1+\alpha)|\varphi(x')|+\alpha\xi^0+\alpha\delta]\nu_\delta+\alpha\delta x^0,$$

where $x'\in[x_\delta, x^0]$. Since $|\varphi(x')|$ and ξ^0 are bounded and $\lim_{\delta\to 0}\nu_\delta=0$, then we can choose δ sufficiently small so that

$$[(1+\alpha)|\varphi(x')|+\alpha\xi^0+\alpha\delta]\nu_\delta+\alpha x^0\delta<\varepsilon.$$

The case $\xi = \xi^0 - \delta$ is treated similarly. Thus, $h(\xi)$ is continuous.

We have thus proved continuity of the function $h(\xi)$ for the case of weak penalties.

For strong penalties, the extremum point of the function f_i is determined by the condition (6), which is solvable for x_i, if $\varphi_i'(0)>\xi_i$, and otherwise $x_i = 0$. This follows from strong concavity of the function $\varphi_i(x_i)$. Therefore at the point x_i we have $0<\varphi_i'(x_i)<\xi_i$. But then by the implicit function theorem, $x_i(\xi_i)$ is continuously differentiable. Thus, the function $h_i(\xi_i)$ is continuous for strong penalties also.

We will now show that $h_i(\xi_i)$ is a decreasing function. First consider the case of weak penalties, when $h_i(\xi_i)=(1+\alpha)\varphi_i(x_i(\xi_i))-\alpha\xi_i x_i(\xi_i)$. Let $\xi_i^2 > \xi_i^1$, then we have $h_i(\xi_i^1) = (1+\alpha)\varphi_i(x_i^1)-\alpha\xi_i^1 x_i^1 \geqslant (1+\alpha)\varphi_i(x_i^2)-\alpha\xi_i^1 > (1+\alpha)\varphi_i(x_i^2)-\alpha\xi_i^2 x_i^2 = h_i(\xi_i^2)$, i.e., $h_i(\xi_i)$ is a decreasing function.

Now assume strong penalties. In this case, $h_i(\xi_i)=\varphi_i(x_i(\xi_i))$, where $x_i(\xi_i)$ satisfies (6). Consider an arbitrary function $h_i(\xi_i)$: $h_i'(\xi_i)=\varphi_i'(x_i(\xi_i))x_i'(\xi_i)$.

Here $\varphi_i'(x_i(\xi_i))>0$. Let us investigate the sign of $x_i'(\xi_i)$. Differentiating the function $\varphi_i(x_i(\xi_i))-x_i(\xi_i)\xi_i$ with respect to ξ_i and equating the derivative to zero, we obtain

$$[\varphi_i'(x_i)-\xi_i]x_i'(\xi_i)=x_i(\xi_i).$$

Since $x_i(\xi_i) > 0$, and $\varphi_i(x_i(\xi_i))-\xi_i<0$, then $x_i'(\xi_i) < 0$. Hence $h_i(\xi_i) < 0$.

b) We have the chain of inequalities

$$\sup_{0<\xi_i<\infty} h_i(\xi_i) \geqslant h_i(0) = \sup_{s_i} \varphi_i(s_i) \geqslant \varphi_i(c).$$

c) We will show that for $\xi_i \to \infty$, $\lim x_i(\xi_i) = 0$. Let $\lim x_i \to c_i^0 > 0$. Then $\varphi_i(x_i)\to\infty$ for $x_i \to c_i^0$, which contradicts boundedness of the derivative φ_i' on (ε, ∞), where $\varepsilon > 0$. Q.E.D.

Proof of Theorem 1. Note that if we take fixed ξ_i for the elements, then the request s_i is selected by each element i so as to maximize its objective function $f_i(\varphi_i(s_i), \xi_i s_i)$ over s_i. Therefore, if the strategies ξ_i selected by the elements determine a Nash equilibrium, the theorem is proved. We will show that the collection of strategies is indeed a Nash equilibrium. Note that for the losers $k\notin Q$ by definition $s_k^* = c$, $w_i^* = \varphi_i(c)$, $f_i(u_i, w_i)=\varphi_i(c)$: for the winners, the payoff is $h_i(\xi)$. Since $h_i(\xi)$ is a continuous decreasing function, the winners $i\in Q$ will choose the least ξ, which is not less than $v_{i_{m+1}}$. The latter follows from the fact that otherwise the element i_{m+1} will be the winner. In this case, the payoff of the element $i\in Q$ will be $\varphi_i(c)\leqslant h_i(v_{i_{m+1}})$. Thus, for any $i\in Q$ the strategy $\xi_i = v_{i_{m+1}}$, maximizes the payoff, i.e., determines a Nash equilibrium. Q.E.D.

Proof of Theorem 4. Sufficiency. Let (13) hold for all $\varphi_i(x)$. Since in the solution of the competition game $\xi(\varphi, x)=\text{const}$, then denoting $\theta[\xi(\varphi, x)] = \lambda$, we obtain

$$d\varphi/dx=\lambda,\ i\in Q. \tag{A.1}$$

These are sufficient conditions of optimality of the solution $\{x_i,\ i\in Q\}$ of the problem (10).

Necessity. Let N_θ be the set of distinguished functions $\varphi_i(x)$, $i = 1, 2, \ldots, n$. For any $v_{i_{m+1}}$ by the assumptions of the theorem, $\{x_i\}$ is an optimal solution of the problem (10). Thus, there exists λ, such that (A.1) holds for any $\varphi_i\in N_\theta$. Therefore the function $\lambda = \theta[v_{i_{m+1}}]$ exists. Noting that in the solution of the competition game $v_{i_{m+1}} = \xi[u_i^Q, x_i^Q]$ for all $i\in Q$, after substituting $\lambda = \theta[\xi(\varphi, x)]$, in (A.1) we obtain (13). Q.E.D.

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