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OPTIMAL ORGANIZATIONAL HIERARCHIES IN FIRMS

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Abstract

One of the topical problems in modern economy is to construct an efficient organizational hierarchy allowing to control the firm with minimal cost. This paper describes the mathematical model of optimal hierarchies in firms. We explore formally such practical effects as optimality of divisional, functional and matrix hierarchy for different levels of standardization, environment instability, functional links intensity, horizontal and vertical integration, etc.

Keywords: model, control, cost, optimal, hierarchy, divisional, functional, matrix, structure.

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Introduction

Any firm is an organization of economic agents (employees)[1]. In the organization employees conform to some rules (mechanisms) regulating their activity and providing the achievement of the general goal of the firm.

The employees in the organization are specialized. Therefore, they are more efficient than the set of self-employed (non-organized) agents. But the employees with different specialization must be coordinated to achieve the general goal. Coordination is a fundamental problem of any organization because activity of a team must be planned and monitored, individual goals must be coordinated, etc. Some organizational hierarchy[2] is created to fulfill the coordination functions (administrative labor) in the firm.

On the one hand, the hierarchy increases efficiency of the employees' interactions (for example, due to the planning and monitoring informational, material and other flows). On the other hand, the performance of coordination (control) functions is costly. In modern economy organizations become increasingly more complex. As a result, the proportion of managers in organizations may exceed 40% (see, for instance, Radner (1992)). So, the key factor of firm's efficiency is the optimality of the hierarchy.

Two-tier hierarchy can be optimal for small firms. In this hierarchy workers on the first (lowest) tier are immediately subordinated to a single manager. As the firm grows, the single manager can not control all interactions between the workers. Therefore, one has to hire several managers to the second tier of the hierarchy and to delegate them the responsibility to control business interactions (flows) within the subordinated groups of workers. But interactions between subordinated groups cause interactions between the managers on the second tier. Several managers on the third tier must control these interactions, etc. In such a way multi-tier hierarchy arises. A superior manager in the hierarchy has an authority over his or her subordinates (managers or

[1] Below we use the terms "organization" and "firm" as synonyms.
[2] The employees on higher tiers of the hierarchy have more authority than the employees on lower tiers. It allows to control the firm even when conflicts between the employees exist.

workers) and a subordinate provides the information to and follows the instructions from his or her superiors.

The design of the hierarchy is one of the aspects of organization design. Management science literature divides the process of organization design (and re-engineering) into three phases[3] (see, for instance, Mintzberg (1979), Williamson (1975)):

I. Technology design: the number of workers, their functions and interaction rules are determined.

II. Hierarchy (organigram) design: the number of managers and their subordination are determined.

III. Mechanism design: superiors' authorities over their subordinates are determined[4].

Typically, an expert in the appropriate field performs the technology design (phase I). There are technological optimization models in different industries, plants, etc. In real firms workers' interaction rules are frequently not formalized. In this case, one can describe the technology using, for example, function modeling methodology (IDEF)[5]. Technological interactions between workers can be mathematically described using a weighted network. The weight of each network link determines intensity of the interaction.

There are many mathematical models of control mechanisms (phase III). Two-tier hierarchy mechanisms (principal-agent problems) have been researched in detail (see, for instance, Hart and Holmstrom (1987), Grossman and Hart (1982 and 1983)). There exist the models of control mechanisms in some types of multi-tier hierarchy (e.g. Melumad, Mookherjee and Reichelstein (1995) explore the delegation mechanism in three-tier hierarchy).

In this paper we concentrate our attention on the phase II. Several papers are focused on the hierarchy optimization problem (phase II) or joint optimization of hierarchy and mechanisms (phases II and III). The

[3] In practice these three phases may not be altogether independent. But it is rather difficult to optimize all these phases at once. To simplify the problem each phase is usually considered separately.

[4] For example, employees' rights and responsibilities are determined.

[5] One defines some aggregated functions (purchasing, production, sales, document processing, etc.) and makes detailed decomposition (fragmentation) down to elementary functions performed by each specific worker. During the decomposition process, one defines interactions between the workers.

study of the hierarchic organizations was pioneered by Simon (1957). His model is based on the following assumptions:

1. The employees on the first (lowest) tier are the only workers performing production labor. All employees on higher tiers are managers performing only administrative labor (control functions).

2. Any employee in the hierarchy has the only immediate superior on the next hierarchical tier. Thus, any hierarchy is a tree. And only employees on adjacent tiers may interact directly.

3. The wage is the same for all employees on one tier. The span of control (the number of manager's immediate subordinates) is the same too. So, employees on one tier are assumed to be identical.

4. The span of control is the same on different tiers of the hierarchy.

5. The wage on the next tier is a constant multiple of the wage on a previous tier. The constant is an exogenous number, which does not depend on the tier and other parameters of the hierarchy.

Williamson (1967) explores a similar model and proves that firm size is limited because of "loss of control" (employees' efficiency decreases from an upper tier to a lower tier). The interlayer efficiency differential is an exogenously given constant. Calvo and Wellisz (1978) explain the wage and the efficiency endogenously. Employee's efficiency depends on his or her wage and the span of control of the immediate superior. The larger the manager's span of control is, the less is his or her subordinates' effectiveness, as individual subordinate is controlled rarely. Using this assumption Calvo and Wellisz (1979) consider the profit maximization model. The profit equals the difference between income (the number of workers multiplied by their effectiveness) and total wages of all employees. In this model both different spans of control and wages on different tiers are possible. Thus, Calvo and Wellisz dispense stringent assumptions 4 and 5 and prove important principles, for example, that in the optimal hierarchy the higher tier the more employee's efficiency and wage per efficiency unit.

Keren and Levhari (1983) optimize the hierarchy's decision-making time[6] (delay on each tier equals the span of control plus con-

[6] Marschak and Radner (1972) study the effect of delay on the value of decisions. This is one of the first models of hierarchy with managers calculating some "decision" (control action).

stant). Average cost per employee is calculated for the hierarchy with minimal decision-making time. This cost allows to calculate the limits of the firm's size. Similar information processing models are explored in numerous papers (see, for example, Van Zandt (1996), Bolton and Dewatripont (1994), Radner (1993)).

Qian (1994) explores Calvo and Wellisz (1979) model by using optimal control techniques, a method pioneered by Keren and Levhari (1979). Continuous approximation is considered (continuous number of employees on each tier). In this case, the optimization problem is simpler than discrete problem[7]. If any employee's effort choice is restricted to only zero or maximal effort, then in Calvo and Wellisz model the optimal employee's wage depends only on the span of control of his or her immediate superior[8]. To maximize profit one has to minimize total wages because employees' efficiency (effort) is maximal. In this case Qian (1994) obtains the optimal hierarchy[9].

Like Qian, in this paper we consider the problem of searching out optimal hierarchy (optimal hierarchy problem), which minimizes total wage of employees (total cost). However, we differ from Qian and other cited above papers in two important respects. First, we consider manager's wage function depending not only on the span of control, but also on sets of workers controlled by the employees immediately subordinated to the manager. So, manager's wage depends on "specificity" and "complexity" of manager's administrative labor (such wage function is called "*sectional*" in this paper). Thus, we do not assume that employees on one tier of the hierarchy are identical. Second, we consider not only tree-like hierarchies, but also more complex hierarchies with multiple subordination or cross-tier subordination[10]. Therefore, we differ from papers, cited above, because we dispense assumptions 2 and

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[7] Van Zandt (1995) examines the validity of continuous approximation of discrete optimal hierarchy problem.

[8] Suppose any employee works at full efficiency or shirks. In this case the employee compares expected loss of wage (the wage multiplied by the loss probability) and shirked time utility. To induce the employee to work efficiently one should calculate such wage that expected loss is greater than or equal to the utility. Loss probability inversely depends on span of control of the immediate superior. Therefore, optimal wage linearly depends on the superior's span of control.

[9] Also Qian (1994) explores more complex cases.

[10] It allows to prove insightful optimality conditions for tree, symmetric tree, etc.

3 (this paper bases only on assumption 1). Optimal hierarchy problem considered in this way is much more complicated. To explore this problem we base on the additional assumption: any hierarchy provides the maximal efficiency of employees. In this case to maximize profit we have to find a hierarchy with minimal total wage (total cost). Thus, control mechanisms (phase III) are not considered and manager's wage (cost) function is given exogenously[11]. We suppose that if employee's wage equals to the cost then his or her efficiency is maximal. Particularly, we entirely abstract from incentive problems[12].

One well-known aspect of hierarchy optimization problem is comparison of divisional, functional and matrix hierarchies[13]. Advantages and disadvantages of these types of hierarchy are often discussed in management science literature (see, for example, Mintzberg (1979)). In a divisional hierarchy, all flows pertaining to a product (region, customer, etc.) are controlled by the divisional managers (for example, a single brand manager for each product). Strategic managers control all flows between different divisional managers. By contrast, in a functional hierarchy, all flows pertaining to a single activity (sales, purchasing, production, etc.) are controlled by the functional managers (for example, sales manager, purchasing manager, production manager, etc.). Strategic managers control all flows between different functional managers. In a matrix hierarchy each worker belongs to one division and one department. For example, if a worker performs marketing functions for the first product then the worker is subordinated to the sales manager and the first product manager. Therefore, divisional and functional managers control all flows; strategic managers perform some strategic functions.

Recently developed models allow to compare mathematically divisional, functional and matrix hierarchies. For example, Maskin, Qian and Xu (2000), Qian, Roland and Xu (1997), Milgrom and Roberts

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[11] In this paper we consider different cost functions. For example, these functions may be defined using technological network (the result of the phase I) and possible controlling mechanisms (the result of the phase III).

[12] It is easy to create incentive mechanism under complete information: costs of maximal efficient employees are compensated and wages of other employees equal to zero. Some incomplete information intensive mechanisms are briefly discussed in the final section.

[13] In several papers terms M-form (multi-divisional form) and U-form (unitary form) are used instead of divisional and functional hierarchy respectively.

(1992) explain mathematically advantages of the divisional hierarchy over the functional hierarchy. Harris and Raviv (2002) develop a model with two "process lines" and two "workers" in each line (for instance, design and marketing sections in Norway and US). In their model all possible hierarchies controlling these four "workers" are compared. But one can not use this approach for larger firms because of huge number of possible hierarchies.

In this paper we show that divisional, functional or matrix hierarchy is optimal for any size of the firm in some circumstances. This fact is important because in general these hierarchies may have much greater costs than an optimal hierarchy. We prove that managers on lower hierarchical tiers must control the most intensive flows because it helps to decrease the strategic managers' costs (Harris and Raviv (2002) prove similar principle). Also we show that if environment stability or standardization[14] decreases then the matrix hierarchy becomes optimal. Thus, the matrix hierarchy is stable with respect to standardization and stability decrease. On the contrary, divisional and functional hierarchies are stable with respect to standardization and stability increase.

With the help of the model introduced in this paper, one can analyze dependences between the type of the optimal hierarchy and horizontal integration (for example, the buying similar firms in other regions), vertical integration (for example, the buying vendors or customers), production volume or functional links intensity change, etc. We prove that the divisional hierarchy is stable with respect to horizontal integration and production volume increase. Vertical integration or functional links intensity increase may cause restructure. On the contrary, the functional hierarchy is stable with respect to vertical integration and functional links intensity increase. Horizontal integration or production volume increase may cause restructure.

Dependences mentioned above take place in many real firms. Many examples without formal proof are considered in management science literature (see, for instance, Mintzberg (1979)). These principles are proved formally in this paper. So, the proposed model explains some effects in real firms.

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[14] In this paper standardization means, for example, job descriptions, products' requirements, common skills and knowledge, etc. Mintzberg (1979) considers different types of standardization in detail.

The optimality of divisional, functional or matrix hierarchy is proved for particular case of sectional cost function[15] introduced in this paper. Sectional functions are also interesting from the mathematical point of view: any additive (with respect to manager's addition) and anonymous (with respect to manager's permutation) hierarchies' cost function can be represented in sectional form (Mishin and Voronin (2003), Mishin (2003b)). In this paper we explore optimization methods that can be used to obtain the optimal hierarchy for numerous classes of sectional cost functions regardless of function's specificity and practical interpretations.

This paper falls into three chapters. Chapter 1 introduces basic definitions and propositions and considers illustrative examples. Chapter 2 proves the optimality of divisional, functional or matrix hierarchy under particular constraints. In Chapter 3 we explore general model with arbitrary sectional cost function and solve optimal hierarchy problem for several cases. The optimization methods introduced in this chapter are used to analyze cost functions corresponding with different types of interactions between manager and immediate subordinates.

Brief summary of this paper and possible extensions of the introduced model are discussed in final section.

All mathematical proofs of the presented below formal statements are published in Mishin (2004c) and can be downloaded using the following stable URL:

http://www.mtas.ru/uploads/optimal_hierarchies_proofs.pdf

[15] Manager's cost depends only on sets of workers controlled by immediate subordinates.

1. Basic Model

In this chapter we define a hierarchy controlling a set of workers. Manager's cost depends on flows between controlled workers. Examples show, how we can model with such cost function several effects that can be observed in real firms.

We can compare costs of different hierarchies, basing on the values of this cost function. If the function conforms to a firm then we can calculate costs of "typical" hierarchies and obtain "the best" typical hierarchy. But it is much more important that we can formulate optimal hierarchy problem. The optimal hierarchy cost is minimal among all possible hierarchies controlling given set of workers. The optimal hierarchy cost may be much less than the cost of the best typical hierarchy. Therefore, it is very useful to obtain optimal hierarchy (solve optimal hierarchy problem). In general this problem is extremely complicated[16]. But in some cases we can solve it using hierarchy optimization methods introduced in this paper.

In the basic model we find optimal hierarchy controlling symmetric process line. This result is used in Chapter 2. More complicated technological networks are not considered in the basic model because it is more convenient to research the general model (Chapter 3).

Sections 1.1-1.6 define optimal hierarchy problem considered in the basic model. These definitions are used below in Chapters 2 and 3 (only cost function type is changed). Results of Section 1.7 allow to exclude non-optimal hierarchies by applying optimal hierarchy conditions. In Section 1.8 sufficient condition of two-tier hierarchy (with single manager) optimality is proven[17]. In Section 1.9 we describe some examples with practical interpretations of the basic model. These examples demonstrate how the model can be used to describe some practical effects in firms. In Sections 1.10 and 1.11 the optimal hierarchy controlling symmetric process line is obtained.

[16] Several similarly problems are described in Goubko and Mishin (2002), Mishin (2004b).

[17] Below in Chapter 3 this condition is generalized for arbitrary sectional functions.

1.1. Workers and Technological Network

Let $N=\{w_1,\dots,w_n\}$ be a set of *workers* who can interact with each other. Let w_{env} be an *environment* interacting with the workers. Typically we denote the workers as $w, w', w'' \in N$.

A *flow function* is a function given by:

$$f:(N \cup \{w_{env}\}) \times (N \cup \{w_{env}\}) \to R_+^p . \qquad (1)$$

Thus, for any pair of workers $w', w'' \in N$ vector $f(w', w'')$ means the *flow intensity* between w' and w'' (p-dimensional vector with nonnegative real components). Each component is an intensity of one type of workers interactions or one type of flow (e.g., material, informational or other type of flow). For example, the first component may denote some material flow and the second one – the flow of information. Thus, the vector $f(w', w'') = (1;0)$ defines material flow of unit intensity and absence of information flow between w' and w''. The vector $f(w', w'') = (2;1)$ may be interpreted as greater flow than (1;0). Thus, a technology defines the flow function f or weighted *technological network f*. For any $w \in N$ the value $f(w_{env},w)$ is a flow between the worker w and the environment.

Flows between workers will be called *flows inside technological network*. Flows between environment and workers will be called *flows between technological network and environment*.

We suppose that the technological network is *undirected* because flow direction is of no importance in our model. Thus, $f(w', w'') = f(w'', w')$ for any $w', w'' \in N \cup \{w_{env}\}$.

There is no *link* between w' and w'' if and only if $f(w', w'') = 0$.[18] So, w' and w'' are *linked* if and only if there are some flows between w' and w''. Also for any $w \in N \cup \{w_{env}\}$ we suppose $f(w,w)=0$ (loop-free network).

For example, consider $N=\{w_1,w_2,w_3\}$ and $p=1$ (there are three workers and only one type of flow). Let the network have four links $f(w_{env}, w_1) = \lambda$, $f(w_1, w_2) = \lambda$, $f(w_2, w_3) = \lambda$, $f(w_3, w_{env}) = \lambda$, where λ is a flow intensity vector. This technological network is shown in Figure 1.

[18] I.e. all vector components equal to zero.

The environment node w_{env} is not shown in the figure. So, the links (w_{env},w_1) and (w_3,w_{env}) are external. This network can be corresponded with process line ("business process"). The worker w_1 gets raw materials from the vendors and executes some production operation. After that the worker w_1 passes the results of his or her operation to the worker w_2. The worker w_2 executes the next production operation and passes the results to the next worker, etc. The last worker (w_3 in Figure 1) executes the last operation and dispatches finished products to the customers.

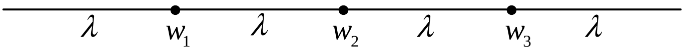

Figure 1. Symmetric Process Line

Such technological network with workers $N=\{w_1,\ldots,w_n\}$ and flows $f(w_{env},w_1)=\lambda$, $f(w_{i-1},w_i)=\lambda$ for each $2\le i\le n$, $f(w_n,w_{env})=\lambda$ will be called *symmetric process line*.[19]

Flows of different intensity are possible in the *non-symmetric process line*. Intensity changes may be caused by the specific nature of interactions at different production stages.

1.2. Managers and Hierarchies

Let M denote a finite set of *managers* who control workers' interactions. Typical managers will be denoted as $m,m',m'',m_1,m_2,\ldots\in M$. Let $V=N\cup M$ denote a set of all *employees* of the firm (workers and managers).

For each manager we need to define his or her subordinates (workers or other managers). Let's define a set of *subordination edges* $E\subseteq V\times M$. Any edge $(v,m)\in E$ means that the employee $v\in V$ is an *immediate subordinate* of the manager $m\in M$. Thus, the edge is directed from the immediate subordinate to the *immediate superior*.

An employee $v\in V$ is *a subordinate* of the manager $m\in M$ (manager m is *a superior* of the employee v), if there exists a path from

[19] All other flows in process line are equal to zero.

v to m. So, there exists such sequence of managers $m_1, m_2, \ldots, m_k \in M$ that the employee v is an immediate subordinate of the manager m_1 ($(v, m_1) \in E$), the manager m_j is an immediate subordinate of the manager m_{j+1} ($(m_j, m_{j+1}) \in E$) for each $1 \leq j \leq k-1$, $m_k = m$. We will say that any superior *controls* his or her subordinates (any subordinate *is controlled* by his or her superiors).

Now we can define the hierarchy formally.

Definition 1. *A directed graph* $H = (N \cup M, E)$ *with a set of managers M and a set of subordination edges* $E \subseteq (N \cup M) \times M$ *is the hierarchy controlling the set of workers N if H is acyclic, any manager has at least one subordinated employee and some manager controls all workers. Let* $\Omega(N)$ *be the set of all hierarchies*.

Acyclicity prevents a "vicious circle". Assume there exists some cycle of managers $m_1, m_2, \ldots, m_k \in M$ ($(m_j, m_{j+1}) \in E$ for each $1 \leq j \leq k-1$, $(m_k, m_1) \in E$). Then each manager is a superior and subordinate of another managers. Such cycle contradicts the main point of the term "subordination". So, Definition 1 ex ante excludes graphs with cycles.

Also Definition 1 excludes the "managers" without subordinates.

According to Definition 1 there exists a manager controlling all workers. Therefore, any set of workers has a common superior and any hierarchy is able to control all workers' interactions.

In Figure 2 there are two examples of the hierarchy over the process line with four workers. The hierarchy a) has "classical" form. Each employee has only one immediate superior (except the top manager, who has no superiors). In Figure 2b) one of the employees has two superiors. Moreover, in Figure 2b) some managers have both immediately subordinated manager and immediately subordinated worker. Such hierarchies may arise in real firms (effects of multi-subordination and interactions between different tiers).

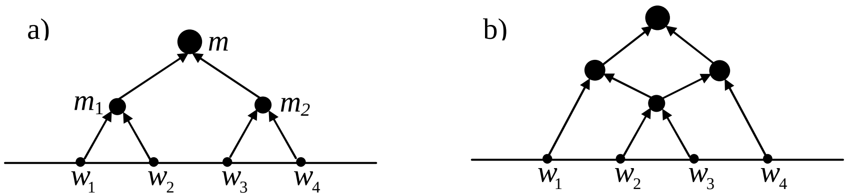

Figure 2. Examples of Hierarchies over the Process Line

1.3. Subordinated Groups of Workers

To define manager's cost we have to formalize his or her labor of managing in the firm (for example, work content). We suppose that administrative (controlling) labor depends on what workers are subordinated to the manager. Below we define group controlled by a manager.

Any nonempty set of workers $s \subseteq N$ will be called a *group of workers*.

Definition 1 implies that in a hierarchy H any manager m has at least one immediately subordinated employee. We can start from a manager m and consider his or her immediate subordinates. After that we can consider their immediate subordinates, etc. Finally we can determine the set of workers subordinated to the manager m. This set $s_H(m) \subseteq N$ is called *manager's m subordinated group of workers*. In other words any manager *m controls the elementary group of workers* $s_H(m)$ in any hierarchy $H \in \Omega(N)$.

Acyclicity implies that any manager has at least one subordinated worker. Therefore, any manager controls non-empty group of workers.

We will leave out inferior index "H" in notation $s_H(m)$ if it is clear what hierarchy we analyze.

It will be convenient to think that any worker $w \in N$ has a subordinated "group" $s_H(w)=\{w\}$ which consists of this worker only. In other words the worker $w \in N$ "controls" the elementary group $s_H(w)=\{w\}$.

In Figure 3 the horizontal plane corresponds with the technological network. A hierarchy is constructed over this plane (network). In Figure 3 the part of hierarchy subordinated to the manager m is shown. This part consists of immediate subordinates of the manager m and his or her subordinates not controlled immediately. In Figure 3 the subordinated group of workers $s_H(m)$ is outlined by ellipse.

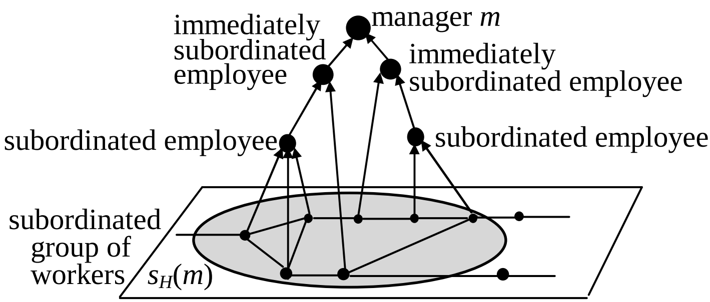

Figure 3. Manager and Subordinated Group of Workers

Consider the simple lemma. It will be necessary further.

Lemma 1. *For any hierarchy H and any manager* $m \in M$ *the equality* $s_H(m) = s_H(v_1) \cup \ldots \cup s_H(v_k)$ *holds, where* $v_1,\ldots,v_k$ *are all immediate subordinates of the manager m. For any employee v subordinated to the manager m the inclusion* $s_H(v) \subseteq s_H(m)$ *holds.*

Let's illustrate the lemma using the example. In Figure 2a) manager m has two immediate subordinates m_1 and m_2. The group of workers $s(m)=\{w_1,w_2,w_3,w_4\}$ is subordinated to the manager m. The groups $s(m_1)=\{w_1,w_2\}$ and $s(m_2)=\{w_3,w_4\}$ are subordinated to the managers m_1 and m_2 respectively. Thus, the group $s(m)$ is divided into the subgroups $s(m_1)$ and $s(m_2)$: $\{w_1,w_2,w_3,w_4\}=\{w_1,w_2\} \cup \{w_3,w_4\}$. In this example subgroups do not overlap. In general case subgroups can intersect (see Figure 2b)).

1.4. Types of Hierarchy, Span of Control

Let's define some types of hierarchy and span of control.

Definition 2. *A hierarchy is a tree, if only one manager m has no superiors and all other employees have exactly one immediate superior. The manager m will be called the root of the tree.*

An example of the tree is shown in Figure 2a). Hierarchy in Figure 2b) is not a tree because one manager has two immediate superiors. Consider one more lemma. It will be necessary further.

Lemma 2. *Consider a hierarchy H, which has only one manager without superiors. Hierarchy H is a tree if and only if any manager's immediate subordinates control non-overlapping groups of workers*.

Thus, in the tree (and only in the tree) the immediate subordinates of any manager do not "duplicate" each other (do not control the same worker).

Definition 3. *A hierarchy is called r-hierarchy if any manager has no more than r immediate subordinates, where* $r > 1$ *is some integer number*. *If r-hierarchy H is a tree then H will be called r-tree.*

The term "span of control" is often used in management science literature. *Span of control* is the maximum number of immediate subordinates, which can be controlled by one manager. If the span of control equals *r* then the hierarchy is *r*-hierarchy.

Lemma 2 implies that immediate subordinates of any manager in a tree control non-overlapping groups. Thus, the maximum number of immediate subordinates equals *n* (if all immediate subordinates are workers). So, the span of control in any tree does not exceed *n*. And *two-tier hierarchy* with single manager controlling all workers (see Figure 4) has the maximal span of control.

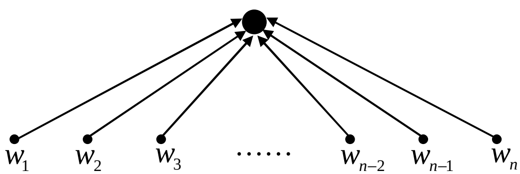

Figure 4. Two-Tier Hierarchy

1.5. Flow Control

In the basic model manager's cost depends on technological flows (technological network). Consider some explanations before the formal definition.

In practice the flow intensity between workers changes with time. But average month or annual intensity is rather stable. Suppose this average intensity does not change. Thus, we suppose that the technological network (function f) is given and fixed. For example, some plant can produce and sell 1000 tons of goods per year. This is the value of the function f (flow intensity). To implement this production volume, managers need to control interactions between the workers.

Each manager controls the flows between his or her subordinated workers. One interpretation of managers' administrative labor is the implementation of some plans. Managers at the top formulate operational plan, which they want to implement. For example, this plan can include day or week sales and purchases volumes, i.e. flows between workers and environment. During the process of disaggregation, managers on each tier add new details to their parts of the plan. For example, to fulfill the sales volume, planned by top manager, the production director can create the plan of production flows. After all tiers of disaggregation workers implement the final detailed plan. At the same time, each manager monitors his or her plan implementation. Thus, each manager controls (e.g., plans and monitors) some flows in the technological network.

Consider some example to explain manager's flows.

Suppose a conflict causes violation of some interaction between workers w_2 and w_3 (see the hierarchy in Figure 5). Thus, actual flow intensity between w_2 and w_3 may be less than necessary flow intensity $f(w_2,w_3)$. The worker w_2 informs the immediate superior m_1 about this interaction problem. The manager m_1 can not solve the problem because the worker w_3 is not subordinated to m_1. Similarly the manager m_2 can not solve the problem after reception of worker's w_3 information. As a result, managers m_1 and m_2 inform their common immediate superior m about the problem. The manager m makes some decision. Managers m_1 and m_2 pass this decision to the workers w_2 and w_3. In such a way the interaction problems (conflicts) are eliminated. Similarly we can consider the planning of the flow $f(w_2,w_3)$. Manager m passes the plan of the flow $f(w_2,w_3)$ to managers m_1 and m_2 who pass the plan to the workers w_2 and w_3 correspondingly. The results of the plan implementation come to the manager m in the reverse order.

Therefore, the manager m controls the flow $f(w_2,w_3)$ and managers m_1 and m_2 participate in this flow control. No managers except the manager m_1 participate in flow $f(w_1,w_2)$ control because m_1 makes all the decisions about this flow independently. Similarly no managers except the manager m_2 participate in flow $f(w_3,w_4)$ control.

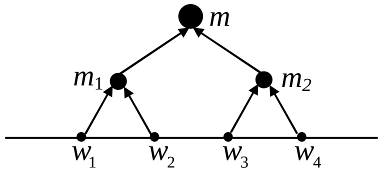

Figure 5. Controlling Tree over the Process Line

Managers m_1 and m participate in the external flow $f(w_{env},w_1)$ control (for example, the purchasing plan is created by the manager m, detailed by the manager m_1 and implemented by the worker w_1). Similarly only managers m_2 and m participate in the external flow $f(w_4,w_{env})$ control.

Thus, managers' m, m_1 and m_2 cost may depend on the following total flows:

m_1: $f(w_1,w_2)+ (f(w_{env},w_1)+ f(w_2,w_3))$,
m_2: $f(w_3,w_4) + (f(w_2,w_3)+ f(w_4,w_{env}))$,
m: $f(w_2,w_3) + (f(w_{env},w_1)+ f(w_4,w_{env}))$.

The example discussed above shows that a manager fulfills "obligations" of two following types:

1. The manager controls such flows within subordinated group that are not controlled by subordinated managers. For example, in Figure 5 the manager m controls the flow $f(w_2,w_3)$.

2. The manager participates in control of the flows between the subordinated group and all other workers, the flows between the subordinated group and the environment. In the expressions given above

these flows are shown in parentheses. For example, in Figure 5 the manager m_1 participates in flows $f(w_{env},w_1)$ and $f(w_2,w_3)$ control.

Let's define manager's "obligations" formally.

Definition 4. *In the hierarchy* $H \in \Omega(N)$ *any manager m operates with the following two types of flows*:

1. *m controls the flows between subordinated workers* $w', w'' \in s_H(m)$ *that are not controlled by subordinated managers. The sum of such flows will be called an internal flow of the manager m and denoted* $F_H^{\mathrm{int}}(m)$;

2. *m participates in control the flows between each subordinated worker* $w' \in s_H(m)$ *and each non-subordinated worker* $w'' \in N \setminus s_H(m)$ *or the environment* $w'' = w_{env}$. *The sum of such flows will be called an external flow of the manager m and denoted* $F_H^{ext}(m)$.

Thus, **a manager controls internal flow and participates in external flow control**. **Total internal and external flows will be called *the flow of the manager***.

The definition implies that the external flow of manager *m* is given by:

$$F_H^{ext}(m) = \sum_{\substack{w' \in s_H(m), \\ w'' \in (N \setminus s_H(m)) \cup \{w_{env}\}}} f(w', w''). \tag{2}$$

The result of the following simple lemma allows to calculate the internal flow.

Lemma 3. *Let $v_1,\ldots,v_k$ be all immediate subordinates of the manager m in the hierarchy H. Then the manager's m internal flow is given by*:

$$F_H^{\mathrm{int}}(m) = \sum_{\substack{\{w', w''\} \subseteq s_H(m), \\ \{w', w''\} \not\subset s_H(v_j) \text{ for each } 1 \le j \le k}} f(w', w''). \tag{3}$$

Thus, we need to sum the flows $f(w', w'')$ inside the group $s_H(m)$, which are not controlled by immediately subordinated managers (i.e. each immediately subordinated manager does not control both w' and w''). In this case (and only in this case) any other subordinated manager does not control this flow, therefore $f(w', w'')$ is included in manager's *m* internal flow.

Thus, for any given *N* and *f* manager's *m* internal and external flows depend only on $s_H(v_1), \ldots, s_H(v_k)$ – the groups of workers controlled by immediate subordinates of the manager *m*.

Definition 1 implies that in any hierarchy *H* there exists a manager *m* controlling all workers. Definition 4 implies that for any workers $w', w'' \in N$ the flow $f(w', w'')$ is controlled either by the manager *m* himself or by his or her subordinated managers. Thus, **any flow inside the technological network is controlled by at least one manager in any hierarchy**.

So, any hierarchy controls all flows. But the number of managers and administrative efforts of each manager differ greatly in various hierarchies. Therefore, it is necessary to find the "best" hierarchy among all hierarchies from $\Omega(N)$. This problem is described formally in the next section.

1.6. Control Cost and Optimal Hierarchy

Each manager bears cost because of flows control. In basic model we assume that manager's cost depends only on total internal and external flows. Let's define it more formally.

Definition 5. *The cost of the manager* $m \in M$ *in the hierarchy* $H \in \Omega(N)$ *is given by*:

$$c(s_H(v_1), \ldots, s_H(v_k)) = \varphi(F_H^{\text{int}}(m) + F_H^{ext}(m)), \qquad (4)$$

where $v_1, \ldots, v_k$ *are all employees immediately subordinated to the manager m,* $s_H(v_1), \ldots, s_H(v_k)$ *are the groups controlled by the employees* $v_1, \ldots, v_k$*,* $\varphi : R_+^p \to R_+$ *is non-decreasing function from* R_+^p *to* R_+.

Thus, the manager's cost is defined by the function $\varphi(\cdot)$ depending on the total manager's flow. The fact that the function $\varphi(\cdot)$ is non-decreasing means that manager's cost does not decrease when one or several flow components rise. In other words, manager's cost does not decrease when "volume" of labor rises. Moreover, manager's cost is non-negative.

Hierarchy's cost equals to total cost of all managers. Optimal hierarchy minimizes this total cost. Let's define it more formally.

Definition 6. *Cost of the hierarchy* $H=(N\cup M,E)\in\Omega(N)$ *equals to total cost of all managers*[20]:

$$c(H)=\sum\nolimits_{m\in M}c(s_H(v_1),\ldots,s_H(v_k))=\sum\nolimits_{m\in M}\varphi(F_H^{\text{int}}(m)+F_H^{ext}(m)),\quad(5)$$

where $v_1,\ldots,v_k$ *are all immediate subordinates of the manager* m.

A hierarchy $H^*\in Arg\min\limits_{H\in\Omega}c(H)$ *with minimal cost will be called the optimal hierarchy*.

Several optimal hierarchies may exist. **This paper focuses on the problem of searching out some optimal hierarchy (optimal hierarchy problem)**. Below we suppose that the set N of workers is given. We need to search out an optimal hierarchy (the number of managers and their subordination) from $\Omega(N)$, which minimizes the cost of control of the workers.

We assume that after searching out an optimal hierarchy it is possible to hire necessary managers and only compensate their costs. So, a manager will control flows if his or her cost is compensated[21] (for example, if we pay out wages). It is clear that we need to know manager's cost to compensate it. Optimal incentive mechanism in a complete information framework is described by Mishin (2004a). This mechanism provides minimal payments, which equal to total managers' costs. Also in that paper some mechanisms have been researched for the case of incomplete information.

Below in this paper we assume that manager's cost function $c(\cdot)$ is known completely[22]. Cost function may be determined directly (for example, using accounting information about manager's cost). Moreover, some "typical" cost functions may be considered (for example, below we analyze power function). We can obtain such function's

[20] In expression (5) $c(\cdot)$ means both manager's cost and cost of total hierarchy.

[21] Some "rate of profit" can be included in the manager's cost function. It is necessary if we need to pay some profit to the managers for their administrative labor.

[22] Cost may consist of manager's wage and any additional cost (for example, wage of maintenance staff).

parameters that function's values have minimal deviations from real managers' costs.

In the basic model manager's cost $c(\cdot)$ depends only on given technological flows[23] and the function $\varphi(\cdot)$. Expressions (2) and (3) imply that internal and external manager's flows depend only on groups controlled by immediately subordinated employees $v_1,\ldots,v_k$. Thus, the manager's cost function (4) depends only on groups $s_H(v_1),\ldots,s_H(v_k)$. Below such functions will be called sectional (see the formal definition on page 79). Therefore, **in basic model we consider an example of sectional cost function**.

It is clear that even in simple cases it is very difficult to find optimal hierarchy using enumerative technique (see Example 1 on page 27). In this paper we develop analytical methods, which help to search out an optimal hierarchy under some restrictions (or methods, which allow to reduce the set of hierarchies containing an optimal hierarchy).

1.7. General Form of Optimal Hierarchy

In this section we prove the proposition, that allows to exclude from consideration certainly non-optimal hierarchies. To prove the proposition we use the following lemma.

Lemma 4. *Let m be any manager in the hierarchy H and* $v_1,\ldots,v_k$ *be all employees immediately subordinated to the manager m. If* $s_H(v_1) \subseteq s_H(v_2)$ *then the following inequality holds*:

$$c(s_H(v_2),\ldots,s_H(v_k)) \leq c(s_H(v_1),\ldots,s_H(v_k)).$$

So, we can remove the subordination edge (v_1,m) *with no manager's cost increase.*

If the group $s_H(v_1)$ is embedded into the group $s_H(v_2)$ then Lemma 4 allows to remove the group $s_H(v_1)$ from arguments with no cost $c(s_H(v_1),\ldots,s_H(v_k))$ increase (with no manager's m cost increase). It may be explained in the following way. The employee v_1 does not control any flow outside the group $s_H(v_2)$. But v_1 can "burden" the

[23] As stated above in the introduction, technological flows can be determined using, for example, function modeling methodology IDEF0.

manager m with interaction problems inside the group $s_H(v_2)$, although all these problems are solved by the employee v_2.

Managers' cost function does not depend on the order of groups $s_H(v_1),\ldots,s_H(v_k)$. Therefore, the lemma holds for any pair of embedded groups. Edge (v_1,m) removal does not change the groups controlled by the managers in the hierarchy. Thus, only manager's m cost can change. The inequality in the lemma leads to no cost increase. So, such edges can be removed with no hierarchy's cost increase. Using this fact we can prove the following important proposition.

Proposition 1. *For any hierarchy* $H_1 \in \Omega(N)$ *there exists such hierarchy* $H_2 \in \Omega(N)$ *that* $c(H_2) \le c(H_1)$ *and the following conditions are satisfied*:

- *(i)* *all employees control different groups of workers*;
- *(ii)* *only one manager has no superiors. All other managers and all workers are subordinated (maybe non-immediately) to this manager*;
- *(iii)* *immediate subordinate of a manager does not control any other immediate subordinate of this manager*.

If H_1 *is r-hierarchy, tree or r-tree then* H_2 *is r-hierarchy, tree or r-tree respectively*.

The proposition is proved using consecutive reconstructions of the hierarchy H_1 with no cost increase. As a result, we obtain the hierarchy H_2 satisfying conditions (i)-(iii). If H_1 is r-hierarchy, tree or r-tree then after a reconstruction we obtain r-hierarchy, tree or r-tree respectively.

The proposition also holds in general model if sectional cost function satisfies the inequality in Lemma 4.

The condition (i) means that there is no pair of managers fully duplicating each other's administrative labor. In other words, there are no managers controlling the same group of workers. In Figure 6a) the example of such duplication is shown. Two managers control the same group of workers $\{w_1,w_2,w_3\}$. We can eliminate one of these managers and subordinate other manager to all immediate superiors of the eliminated manager. The cost of the hierarchy does not increase after this reconstruction. Particularly, the condition (i) leads to the fact that **any manager has at least two immediate subordinates** (otherwise Lemma

1 implies that the manager and his or her only immediate subordinate control the same group of workers).

The condition (ii) means that exactly one manager *m* has no superiors. This manager controls all workers ($s_{H_2}(m) = N$) and all other managers in the hierarchy. The manager *m* will be called *top manager*.

So, the condition (ii) corresponds to the practice of organization design: there is one and only one top manager whose decisions must be implemented by all other managers and workers (for example, the top manager can eliminate a conflict between any set of employees in the firm). In Figure 6b) there are two managers with no superiors. So, the condition (ii) is violated. Obviously the “redundant” manager can be removed with no cost increase.

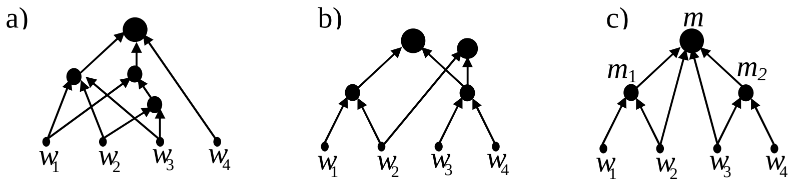

Figure 6. Hierarchies a)-c) Violate Conditions (i)-(iii) Respectively

The condition (iii) can be interpreted as follows. Assume the manager m_1 is immediately subordinated to the manager *m*. Then *m* does not immediately control the subordinates of the manager m_1. The condition corresponds with “normal” activity of the firm, when any manager controls subordinated employees only by means of his or her immediate subordinates, but not directly. In Figure 6c) the top manager *m* directly controls the workers w_2 and w_3, although these workers are also controlled by subordinated managers m_1 and m_2. Lemma 4 implies that edges (w_2,m) and (w_3,m) can be removed with no cost increase.

Proposition 1 implies that **there exists optimal hierarchy satisfying conditions (i)-(iii)**.[24] It simplifies optimal hierarchy problem because we can ignore hierarchies that violate condition (i), (ii) or (iii).

[24] If H_1 is optimal hierarchy then Proposition 1 leads to the hierarchy H_2 satisfying conditions (i)-(iii). And cost of H_2 is less than or equal to cost of H_1. Therefore, H_2 is optimal hierarchy.

Moreover, Proposition 1 leads to the following fact. **If there exist optimal *r*-hierarchy, tree or *r*-tree then there correspondingly exist optimal *r*-hierarchy, tree or *r*-tree satisfying conditions (i)-(iii)**.

All optimal hierarchies obtained in this paper satisfy conditions (i)-(iii).

1.8. Two-Tier Hierarchy Optimality Condition

Consider sufficient condition of two-tier hierarchy optimality in basic model.

Proposition 2. *Let the cost function* $\varphi(\cdot)$ *be subadditive, i.e. for any* $x, y \in R_+^p$ *the inequality* $\varphi(x+y) \le \varphi(x) + \varphi(y)$ *holds. Then two-tier hierarchy with single manager is optimal*.

The subadditivity condition means that cost $\varphi(x+y)$ of one manager controlling total flow $x+y$ does not exceed cost $\varphi(x)+\varphi(y)$ of two managers controlling parts x and y of the total flow. In this case the optimal hierarchy consists of the single manager controlling all flows between the workers. This manager's cost is less than or equal to total cost of managers in any other hierarchy.[25]

Lemma 5. *For one-dimensional flows* ($p=1$) *concave cost function* $\varphi(\cdot)$ *is subadditive*.

Lemma 5 and Proposition 2 imply that **cost function concavity leads to the optimality of two-tier hierarchy** if all flows in the technological network have the same type (flow intensity vector has only one component). For multi-component flows this is incorrect. Below we consider an example of concave cost function and optimal hierarchy with individual manager controlling each type of flow. This hierarchy is optimal due to the managers' specialization (division of managers' labor, see Example 3 on page 31).

[25] In Section 3.3 the subadditivity condition is generalized for arbitrary sectional functions. Using this generalization we obtain optimality conditions for two contrary types of hierarchy: two-tier hierarchy with single manager and 2-hierarchy with maximal number of managers.

Two-tier hierarchies (so called “simple” hierarchies, Mintzberg (1979)) prevail in small firms. But when the firm grows the total flow increases. When the total flow is sufficiently large, single manager is overloaded. Thus, he or she has to hire “assistants”, the hierarchy becomes multi-tier hierarchy. For example, in Section 1.11 for power cost function we find the optimal hierarchy controlling symmetric process line. In this case we prove that optimal hierarchy has many tiers for sufficiently large firm (with large number of workers).

1.9. Some Illustrative Examples

Consider some examples to illustrate the basic model of the optimal hierarchy.

Example 1. The expediency of multiple-subordination for asymmetric process line. Consider the asymmetric process line with four workers and the following flows: $f(w_{env},w_1)=3$, $f(w_1,w_2)=1$, $f(w_2,w_3)=5$, $f(w_3,w_4)=1$, $f(w_4,w_{env})=3$. Consider manager’s cost function $\varphi(x)=x^3$, where x is the value of the manager’s flow. For this example the optimal hierarchy H is shown in Figure 7. The manager m_1 has two immediate superiors. So, there exists optimal hierarchy with multiple-subordination.

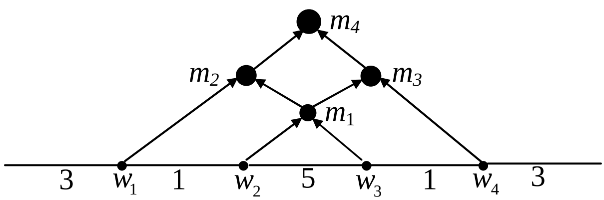

Figure 7. An Example of Optimal Hierarchy over Asymmetric Process Line

Let’s calculate the flow and the cost of each manager:

m_1: $c(\{w_2\},\{w_3\})=\varphi[F_H^{\text{int}}(m_1)+(F_H^{ext}(m_1))]=[5+(1+1)]^3=343$;

m_2: $c(\{w_1\},\{w_2,w_3\})=\varphi[F_H^{\text{int}}(m_2)+(F_H^{ext}(m_2))]=[1+(3+1)]^3=125$;

m_3: $c(\{w_4\},\{w_2,w_3\})=\varphi[F_H^{\text{int}}(m_3)+(F_H^{ext}(m_3))]=[1+(1+3)]^3=125$;

m_4: $(\{w_1,w_2,w_3\},\{w_2,w_3,w_4\})=\varphi[F_H^{\text{int}}(m_4)+(F_H^{ext}(m_4))]=[0+(3+3)]^3=216$.

Thus, the total cost of the hierarchy equals:

$c(H)= c(\{w_2\},\{w_3\})+ c(\{w_1\},\{w_2,w_3\})+ c(\{w_4\},\{w_2,w_3\})+$
$+c(\{w_1,w_2,w_3\},\{w_2,w_3,w_4\})=343+125+125+216=809$.

Let's prove that the cost $c(H)=809$ can not be further diminished. Let H^* be any optimal hierarchy satisfying conditions (i)-(iii) of Proposition 1. In the hierarchy H^* there exists at least one manager m with no subordinated managers.

If m controls three or more workers then the manager's m flow is greater than or equal to 10. So, this manager's cost is greater than or equal to 1000 (manager's cost is more than $c(H)=809$). Therefore, m should control exactly two workers in the optimal hierarchy.

If m controls non-consecutive workers in the process line (for instance, w_1 and w_3) then $F_H^{\text{int}}(m)=0$ (m only participates in external flow control but does not control internal flows). In this case we can eliminate m and subordinate the workers from $s_{H^*}(m)$ to the immediate superiors of the manager m with no change of their costs. It contradicts to the optimality of the hierarchy H^*. So, manager m in the optimal hierarchy may control only consecutive workers in the process line.

If manager m controls workers w_1 and w_2 (or w_3 and w_4) then manager's m cost equals $9^3=729$. Moreover, the top manager participates in control of the flows between the technological network and the environment. So, his or her cost is greater than or equal to $6^3=216$ and the inequality $c(H^*)>729+216=945$ holds. It contradicts to the optimality of the hierarchy H^*. Thus, in the hierarchy H^* there is a single manager m on the second tier (with no subordinated managers). m immediately controls workers w_2 and w_3. Therefore, the manager m controls the maximal flow $f(w_2,w_3)=5$.

The example illustrates the following general principle: **flows with maximal intensity must be controlled by the managers on low tiers of the hierarchy**. This principle is well known in real firms and described in management science literature (see, for instance, Mintzberg (1979)). In the example we consider extreme case, when maximal flow must be controlled by special manager m on the second tier.

The manager m is the only manager on the second tier. Therefore, m is subordinated to all other managers in the hierarchy.[26] Then the workers w_2 and w_3 are immediately subordinated only to the manager m because otherwise the condition (iii) of Proposition 1 is violated. So, after we hire the manager m we have to construct the optimal hierarchy H^* over three employees: w_1, m, w_4. Besides H (see Figure 7) there exist three such hierarchies satisfying conditions (i)-(iii) of Proposition 1. These hierarchies are shown in Figure 8.

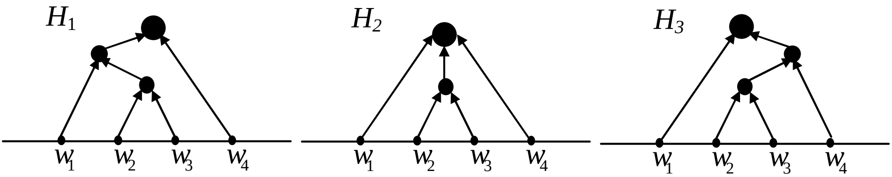

Figure 8. Non-Optimal Hierarchies over Asymmetric Process Line

It is easy to calculate costs $c(H_1)$=811, $c(H_3)$=811, $c(H_2)$=855. All hierarchies in Figure 8 are non-optimal because $c(H)$=809. Thus, the hierarchy H=H^* is the only optimal hierarchy[27].

One interesting questions discussed in this paper is the optimality of trees. Tree is a typical hierarchy for many real firms. Example 1 shows that in some cases the minimal cost tree is non-optimal. Thus, **in some cases there does not exist optimal hierarchy among the trees**. Below we prove the optimality of the tree for the symmetric process line (see Section 1.10). Moreover, in Section 3.2 we consider sufficient tree optimality condition. If the optimal tree exists then we can find the optimal hierarchy using the algorithms of searching minimal cost tree (Mishin and Voronin (2001, 2003)). These algorithms are described briefly in Section 3.2.

[26] Consider some manager $m' \neq m$. Manager m' has an immediately subordinated manager m'' (otherwise m' is a manager on second tier, i.e. $m' = m$). If $m'' \neq m$ then m'' has an immediately subordinated manager too and we can repeat reasoning. In such a way we can construct the path from m to m'. So, the manager m is subordinated to any other manager m'.

[27] We consider only hierarchies satisfying the conditions (i)-(iii) of Proposition 1.

Example 2. Firm growth with control cost decrease. Consider the asymmetric process line with four workers, the flows $f(w_{env},w_1)=1$, $f(w_1,w_2)=5$, $f(w_2,w_3)=1$, $f(w_3,w_4)=5$, $f(w_4,w_{env})=1$ and the manager's cost function $\varphi(x)=x^2$, where x is the value of the manager's flow.

To start with we suppose that the technological network $N=\{w_2,w_3\}$ consists only of workers w_2 and w_3. So, workers w_1 and w_4 are not part of the firm (for example, the vendor and customer). Then there exists the only hierarchy that satisfies conditions (i)-(iii) of Proposition 1 (page 24). This hierarchy is shown in Figure 9a).

Assume we can extend the firm by adding workers w_1 and w_4. This extension can be interpreted as follows. For example, large wholesale company buys the production firm (the "worker" w_1) and the chain of shops (the "worker" w_4) to control all the stages from production to the ultimate consumer. Large flow $f(w_1,w_2)=5$ may be caused by purchasing problems, e.g. large quantity of defective goods. Similarly the large flow $f(w_3,w_4)=5$ may be caused by some selling problems, e.g. customers often return defective goods.

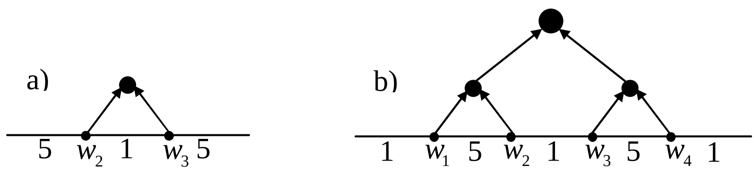

Figure 9. Firm Growth with Control Cost Decrease

Thus, after the extension the firm controls the whole technological network $N=\{w_1,w_2,w_3,w_4\}$. So, we can reconstruct the hierarchy as shown in Figure 9b). We can hire two managers on the second tier and give them the responsibility to control the greatest flows $f(w_1,w_2)=5$ and $f(w_3,w_4)=5$. Let's compare costs of hierarchies in Figures 9a) and 9b):

a) $(5+1+5)^2=121$,

b) $(1+5+1)^2+(1+5+1)^2+(1+1+1)^2=49+49+9=107$.

So, control cost can decrease with the technological network growth (including new workers, which were part of the environment). It could be a reason to buy some unprofitable business because it can reduce cost of control of the main business. Such facts often occur in practice. For example, in ninetieth years of the XX century many Russian food plants were transformed in vertically integrated companies by acquisition of farms in the corresponding region. These farms were unprofitable but provided regular supplies of cheap raw materials (see, for example, Khramova and Wehrheim (1997)).

Example 3. Multi-component flows. Lemma 5 and Proposition 2 imply that two-tier hierarchy is optimal for concave cost function and one-dimensional flows. Below we show that this lemma is not valid for multi-dimensional flows. Consider two-dimensional flows (p=2). The first flow component corresponds to the material flows. The second flow component corresponds to the informational flows. The technological network $N=\{w_1,w_2,w_3,w_4\}$ is shown in Figure 10.

The worker w_1 obtains raw materials from the vendors and produces some components. After that the worker w_1 passes these components to the worker w_2. The worker w_2 assemblies the components and dispatches finished product to the customers. Flow intensity may depend on the number of different material types. So, the worker w_1 gets raw materials of the one type and produces components of three types. The worker w_2 assembles these components and dispatches the finished product of the one type. Thus, material flow $f(w_1,w_2)$ inside the technological network is greater than material flows $f(w_{env},w_1)$ and $f(w_2,w_{env})$ between the technological network and the environment.

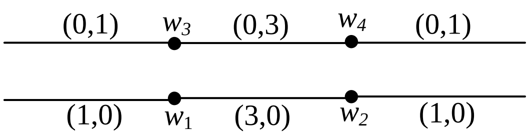

Figure 10. Example of Technological Network
with Two-Component Flows

The worker w_4 negotiates with customers, prepares and concludes contracts, accounts for the payments, shipments, etc.

The worker w_4 passes the information about the required production volume to the worker w_3. Using this information the worker w_3 generates raw materials orders, accounts for the raw material procurement, makes payments, etc. Also the worker w_3 can pass information (required to calculate price and time of delivery) to w_4.

Informational flow $f(w_3,w_4)$ inside the technological network is greater than informational flows $f(w_{env},w_3)$ and $f(w_4,w_{env})$ between the technological network and the environment. For example, the number of internal documents can be significantly exceed the number of external documents.

Assume manager's cost function is given by $\varphi(x,y)=\sqrt{x}+\sqrt{y}+\sqrt{xy}$, where (x,y) is vector of total manager's flow. The function is concave. For example, a few efforts are required from the manager and his or her flow increase decreases the marginal cost[28]. The item $\sqrt{xy}$ may correspond to the manager's specialization. This item equals to zero if the manager controls flows of one type (for instance, the manager controls either production or documents circulation). In this case the manager specializes in particular area of controlling labor. So, the manager can control the flows effectively and the controlling cost is minimal. If the manager controls the heterogeneous flows then his or her efficiency decreases because of non-specialized controlling labor. So, the cost of such manager increases. Thus, the division of controlling labor decreases manager's cost.

Consider the two-tier hierarchy H_1 shown in Figure 11a). There is a single manager in H_1. His or her total flow equals (5,5). So, hierarchy's cost equals $c(H_1)=\varphi(5,5)=2\sqrt{5}+5$.

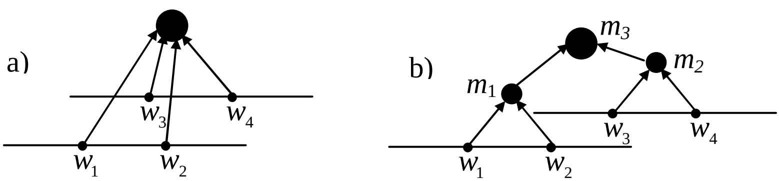

Figure 11. a) Non-Optimal Two-Tier Hierarchy,
b) the Hierarchy with Specialized Managers m_1 and m_2

Consider the hierarchy H_2 with tree managers. This hierarchy is shown in Figure 11b). The manager m_1 controls the production only. So,

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[28] Or manager's cost per unit flow.

m_1 controls workers w_1 and w_2. Manager's m_1 flow equals (5,0) and his or her cost equals $\varphi(5{,}0)=\sqrt{5}$. Similarly the manager m_2 controls the documents circulation (informational flows). So, m_2 controls workers w_3 and w_4. Manager's m_2 flow equals (0,5) and his or her cost equals $\varphi(0{,}5)=\sqrt{5}$. Managers m_1 and m_2 are subordinated to the top manager m_3. The manager m_3 participates in control of heterogenous flows $f(w_{env},w_1)$, $f(w_2,w_{env})$, $f(w_{env},w_3)$, $f(w_4,w_{env})$ between the technological network and the environment. So, m_3 only participates in control of relationships with customers and vendors because all flows inside the technological network are controlled by subordinated managers m_1 and m_2. Thus, manager's m_3 cost equals $\varphi(2{,}2)=2\sqrt{2}+2$. The hierarchy's cost equals $c(H_2)=2\sqrt{5}+2\sqrt{2}+2$.

We can see that the inequality $c(H_2)<c(H_1)$ holds. Therefore, **if the flows in the technological network have several components then the cost of two-tier hierarchy can be reduced by hiring of several specialized managers** (even with concave cost function).

Above examples show that using basic model we can mathematically describe some effects in real firms. But these examples also show that optimal hierarchy problem is very complicated. In this chapter we solve this problem only for particular case – symmetric process line.[29]

1.10. Optimal Hierarchy Controlling Symmetric Line

Consider optimal hierarchy problem for the symmetric process line (see, for example, Figure 1 on page 13). This is the simplest technological network. Some flows move along the line. For example, the first worker gets raw materials, executes some production operation and passes semi-finished products to the second worker in the line. Similarly the material flow moves further down to the last worker in the line. The last worker dispatches finished products to customers. In addition to the material flows the process line may include

[29] This result will be used below in Chapter 2. Optimization methods for more complicated cases are described in Chapter 3 for arbitrary sectional cost function. Therefore, we do not consider such methods in basic model.

informational flows or flows of other types. In this case each flow in the process line consists of multiple components (p>1).

In contrast to arbitrary technological network in symmetric process line the following important conditions hold.

1. The flows processing is sequential. So, each worker interacts only with the previous worker and the next worker in the line.

2. The flow intensity is the same in all production phases. So, interactions intensity does not change along the line.

In practice conditions 1 and 2 may be violated. Different technological routes are possible. Defective goods may be returned to the first worker in the line for revision. The flow intensity can increase and decrease (for instance, it is very simple to control the raw material of the same type in the first phase of the process line, but it is hard to control many components in the middle phases of the process line). However, in many cases the technological network is similar to the symmetric process line. In this section we find the optimal hierarchy for such a network. If technological network is more complex then we can use the other methods described in Chapter 3 to find the optimal hierarchy.

Now we consider the set of workers $N=\{w_1,\ldots,w_n\}$ with flows $f(w_{env}, w_1) = \lambda$, $f(w_{i-1}, w_i) = \lambda$ for each $2 \le i \le n$, $f(w_n, w_{env}) = \lambda$.

Proposition 3. *There exists the optimal tree H controlling symmetric process line and satisfying the following conditions*:

1. the cost of H is less than or equal to total cost of any managers controlling all flows inside the process line[30];

2. in the hierarchy H any manager controls the group of consecutive workers in the process line;

3. if cost function is convex then in the hierarchy H the numbers of immediate subordinates of all managers are equal or differ by one.

Proposition 3 implies that **we can search the optimal hierarchy among the trees** (non-tree hierarchies can be excluded). Example 1 described above shows that this conclusion is not true for asymmetric process lines.

[30] In other words cost of *H* is less than or equal to total costs of any managers controlling all flows, even if these managers are not a hierarchy (even if the top manager controlling all workers does not exist).

Let's explain condition 1 of Proposition 3. In any hierarchy $\Omega(N)$ managers control all flows inside the technological network. However, some managers can control all these flows, even if these managers are divided into several hierarchies (each hierarchy can control only the part of the process line). For example, in Figure 12 some decentralized structure without single top manager is shown. In this structure the managers m_1 and m_2 control different parts of the process line, but m_1 and m_2 have no common superior. Such decentralized structures[31] are used in real firms very seldom because of lack of the top manager authorized to control any employees in the firm.

Proposition 3 leads to the fact that cost of the tree H is less than or equal to total cost of any managers controlling all flows inside symmetric process line. Therefore, **cost of optimal tree is less than or equal to the cost of any decentralized structure without single top manager**.

Definition 1 implies that any hierarchy has the top manager (has centralized control). Optimal hierarchy minimizes the cost among all structures controlling all flows inside symmetric process line[32].

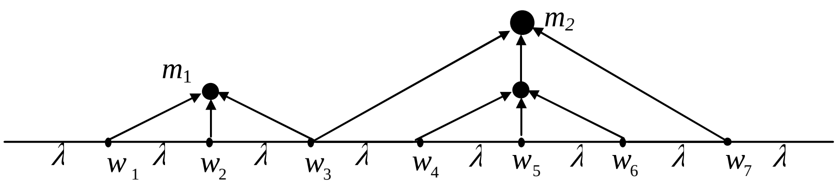

Figure 12. Example of Decentralized Structure without Single Top Manager

Let's explain condition 2 of Proposition 3. This condition implies that we can restrict our attention only by the trees in which every manager controls consecutive workers in the process line. For example, we

[31] In this paper centralized control (centralized structure) means that there exists the top manager control of all the employees in the firm, but does not mean that this manager has all controlling authorities.

[32] This result is used below in Chapter 2 to construct optimal hierarchy controlling several process lines with functional links.

can exclude hierarchies where some manager controls the group $\{w_1,w_2,w_4\}$. Consider Figure 13. Let's subordinate the worker w_4 to the manager m_1 instead of the worker w_3. And vice versa let's subordinate to the manager m_2 the worker w_3 instead of the worker w_4. Then we obtain the following results:

1. The costs of managers m_1 and m_2 increase.
2. The manager m_1 does not control the flow $f(w_2,w_3)$, but only participates in control[33]. Similarly the manager m_2 does not control the flow $f(w_4,w_5)$, but only participates in control. Thus, the superior manager m_3 has to control these flows. So, the value $f(w_2,w_3)+f(w_4,w_5)$ is added to manager's m_3 flow. Therefore, manager's m_3 cost increases.

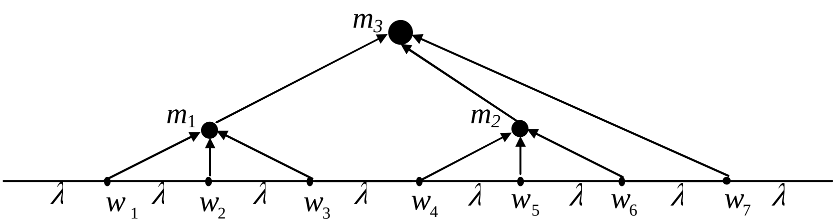

Figure 13. Example of Hierarchy Controlling Symmetric Process Line

This example illustrates the result of Proposition 3: the subordination of non-consecutive workers in the process line increases the cost of hierarchy. The interpretation of this property is evident. **Each manager has to control one *part of the process line***. If we try to subordinate different parts to one manager then hierarchy's cost increases and the hierarchy becomes non-optimal.

In Figure 13 the manager m_1 controls the group $\{w_1,w_2,w_3\}$ with three workers. There are flows $f(w_{env},w_1)$ and $f(w_3,w_4)$ between the group $\{w_1,w_2,w_3\}$ and the other workers and the environment. Thus, after hire of the manager m_1 he or she can be viewed as a worker from the point of view of superior managers. As the manager m_1 has three subordinated workers, the length of the process line is "reduced" by two elements

[33] After described reconstruction this flow is external for the manager m_1.

because three workers are "replaced" by one manager. The manager m_2 "reduces" the length of the process line again. Instead of three workers we can subordinate the manager m_1 and two workers to the manager m_2. In this case the hierarchy's cost does not change. But similar subordination causes the increase of the number of the hierarchical tiers. Therefore, preferable hierarchy is shown in Figure 13.

If a manager m in the tree H has k immediate subordinates then the group $s_H(m)$ is divided into k subgroups (controlled by the immediate subordinates). So, some part of the process line is divided into k "subparts". Then the manager m controls k–1 internal flows and participates in control of two external flows. Therefore, if in the tree any manager controls the group of consecutive workers in the process line, then the cost of the manager with k immediate subordinates is given by:

$$\varphi((k+1)\lambda)\,. \tag{6}$$

If we subordinate some number of workers r_1 to the manager m_1, then the process line is "reduced" by $r_1 - 1$ (r_1 workers are "replaced" by one manager). Similarly we can hire the manager m_2 and subordinate to him r_2 workers or managers not subordinated yet, etc. Finally we hire the top manager m_q[34] to control all the process line. He or she is the single manager without superiors. So, the process line is "reduced" to one manager m_q. Thus, the equality $n-(r_1-1)-(r_2-1)-...-(r_q-1)=1$ holds. Using this equality we obtain the following constraint on number of immediate subordinates of managers in any tree:

$$r_1+\ldots+r_q=n+q-1. \tag{7}$$

Formula (6) implies that managers' costs equal $\varphi((r_1+1)\lambda),\ldots,\varphi((r_q+1)\lambda)$. To obtain optimal hierarchy we need to solve only the following optimization problem:

$$\varphi((r_1+1)\lambda)+\ldots+\varphi((r_q+1)\lambda)\to\min\,, \tag{8}$$

with constraints (7), $r_1,\ldots,r_q\geq 2$, $1\leq q\leq n-1$.

Thus, **for symmetric process line optimal hierarchy problem is reduced to constrained optimization problem with criterion function (8) depending on *q* integer variables** (such problem must be solved for each q). To solve the problem (8) we can use classical dis-

[34] q is total number of managers in the hierarchy.

crete optimization methods or algorithms of minimal cost tree search (Mishin and Voronin (2001, 2003)). These algorithms obtain the tree with minimal cost for arbitrary sectional function[35].

Proposition 3 implies that the problem (8) can be solved analytically for convex cost function. In optimal tree the numbers of immediate subordinates of all managers are equal or differ by one. So, numbers $r_1,\dots,r_q$ are equal or differ by one[36]. Let r be minimal number of immediate subordinates of a manager. Then any manager has either r or $r+1$ immediate subordinates. Let $q_1>0$ be the number of managers with r immediate subordinates. Then $q_2=q-q_1$ is the number of managers with $r+1$ immediate subordinates. Left-hand member in expression (7) is given by $q_1 r + q_2(r+1) = qr + q_2$. Therefore, the following equalities hold:

$$qr + q_2 = n + q - 1,\ r = \lfloor (n+q-1)/q \rfloor .^{37} \qquad (9)$$

If $n-1$ contains q (the residue of division $n-1$ by q equals to zero) then $q_2=0$ and all managers have the same number $r=(n+q-1)/q$ of immediate subordinates. Otherwise formula (9) leads to the fact that r is floor, q_2 is residue of division $n-1$ by q.

Thus, if cost function is convex and q is fixed then formula (9) allows to calculate the numbers $r_1,\dots,r_q$ and formula (8) allows to calculate the cost of the tree. Therefore, to solve optimal hierarchy problem we have to find only the optimal number of managers $1 \le q \le n-1$. It can be done in $n-1$ steps. In other words, **for any convex function we can obtain the optimal tree controlling symmetric process line by comparing costs of *n*–1 trees**. If $q=1$ then the tree is two-tier hierarchy with maximal number of immediate subordinates $r=n$. If $q=n-1$ then the tree is 2-hierarchy with minimal number of immediate subordinates $r=2$.

In the following section we consider important particular case (power cost function). For this case optimal q and r can be found ana-

[35] Algorithms are briefly described in Section 3.2.

[36] If two numbers differ by two or more then in the proof of Proposition 3 maximal number decreases and minimal number increases with no increase of cost of the tree.

[37] The formula $\lfloor (n+q-1)/q \rfloor$ means floor of the number $(n+q-1)/q$ (maximal integer is less than or equal to $(n+q-1)/q$).

lytically (there exists optimal span of control r_*, which does not depend on n and λ).

1.11. Optimal Span of Control for Power Cost Function and Symmetric Line

In economic science power cost functions are used often. One of well-known functions is quadratic cost function. Below in this section and in Chapter 2 we obtain optimal hierarchy for power cost function that depend on manager's total flow.

Consider symmetric process line (see, for example, Figure 1 on page 13) with one-dimensional flows. Flow intensity in the line equals to some non-negative value $\lambda \geq 0$ (for example, material flow intensity). Then we can consider *manager's power cost function*:

$$\varphi(x) = x^{\alpha}, \tag{10}$$

where x is non-negative total manager's flow[38], $\alpha \geq 0$ is an exponent.

The exponent α will be interpreted as *environment instability*.

Consider an example, when manager's cost depends on instability. Suppose the firm produces only one modification of a product in a stable environment[39]. If market capacity is limited and unstable then the firm has to produce several modifications of the product and requirements may change permanently. Suppose the required number of modifications equals to the environment instability α. Manager's total flow x may be interpreted as number of components, whose production is controlled by the manager. The manager must decide how many components should be used to produce each modification. Thus, the manager must decide that $0 \leq x_1 \leq x$ components should be used to produce the first modification, $0 \leq x_2 \leq x$ – to produce the second modification, etc. There are many options of choosing numbers $x_1, \ldots, x_\alpha$. Order of

[38] For example, if manager controls and participates in control of k flows then $x = k\lambda$.

[39] For example, in this case production of one modification may be most profitable (if production capacity is limited and market capacity is unlimited).

greatness of these number of options equals $x^{\alpha-1}$.[40] To analyze each option manager has to calculate, for example, the cost of production of each from x components taking into account technological constraints[41]. Therefore, labour intensity of optimal option choice may grow[42] as x^{α}. So, this value may correspond to the manager's cost.

Considered example illustrates that manager's cost can be modeled using function (10), where the exponent α corresponds to the environment instability. In the example the exponent α equals to the number of modifications demanded by unstable market. In many practical cases the exponent α may be less than number of modifications because the most of modifications may be similar in respect to manager's cost (for example, costs of manufacture and production techniques of some modifications may be equal). Moreover, manager's cost may depend on over instability factors (fluctuation of personnel, raw materials quality, fluctuation of vendors, etc.). Therefore, we suppose that the exponent $\alpha > 1$ equals to some generalized index of environment instability and manager's cost is determined by (10).[43] We suppose values $\alpha \leq 1$ correspond with stable environment.

[40] For example, if $\alpha = 2$ then there exists x+1 option of choosing the number x_1. After choosing x_1 the number x_2 can be calculated using the equality $x_1+x_2=x$. If $\alpha = 3$ then number of options equals $x^2/2+3x/2+1$. Therefore, the order of greatness is equal to $x^{\alpha-1}$ again, etc.

[41] For example, components for one modification must be produced only by some lots but not one by one. If we produce some number of components not divisible by size of the lot then costs of manufacture increase (the profit decreases). Therefore, "optimal" choosing of numbers $x_1,\ldots,x_\alpha$ may depend on different technological factors.

[42] At the beginning of some planning period (for example, at the beginning of the month) the manager must plan production of such modifications that have never been produced before (because of market instability). Therefore, optimal plan (numbers $x_1,\ldots,x_\alpha$) may be unknown. Simple methods of optimal plan search may be unknown too. So, the managers must analyze all possible options.

[43] So, we suppose that instability increase causes α increase and vice versa. Also the exponent α may depend on many other factors (for example, personal abilities of the manager or such factors as environment "complexity", "hostility" described in management science literature). But below in this paper the exponent α will be interpreted as environment instability.

Thus, in stable environment with $\alpha = 1$ manager's cost grows linearly if the flow controlled by the manager increases (each additional flow unit causes the same cost). If $\alpha > 1$ then environment instability causes convexity of the cost function (each additional flow unit causes increasing cost).

If $\alpha \le 1$ then cost function (10) is concave. Therefore, Proposition 2 and Lemma 5 lead to the fact that two-tier hierarchy with single manager is optimal in a stable environment.

Let's obtain optimal hierarchy controlling symmetric process line in unstable environment.

Consider only trees satisfying the following conditions: any manager controls one part (the group of consecutive workers) of the process line and the numbers of immediate subordinates of all managers are equal or differ by one. Proposition 3 implies that there exists optimal hierarchy among such trees because power cost function is convex for $\alpha > 1$.

If some manager has r immediate subordinates then for power function formula (6) leads to the following expression for the manager's cost:

$$\varphi((r+1)\lambda) = (r+1)^{\alpha}\lambda^{\alpha}. \tag{11}$$

If n–1 is divisible by number of managers q in optimal tree, then formula (9) leads to the number of immediate subordinates $r = 1 + (n-1)/q$ of any manager. In this case total cost of all managers in the tree is given by:

$$(r+1)^{\alpha}\lambda^{\alpha}(n-1)/(r-1), \tag{12}$$

where $(n–1)/(r–1)$ is the number of managers.

Expression (12) allows to suppose that optimal hierarchy can be found by choosing of optimal span of control r_*, which minimizes $(r+1)^{\alpha}/(r-1)$. The following proposition confirms this hypothesis.

Proposition 4. *For symmetric process line with one-dimensional flows and power cost function with $\alpha > 1$ the optimal span of control r_* equals to one of two integer numbers closest to the value $(\alpha+1)/(\alpha-1)$.*

If $n-1$ *is divisible by* r_*-1 *then* r_**-tree* H^* *is the optimal hierarchy. In* H^* *each manager controls one part of the process line and has exactly* r_* *immediate subordinates. Cost of the tree* H^* *equals* (12) *with* $r=r_*$*. For arbitrary n formula* (12) *gives a lower bound of cost of control of all flows inside symmetric process line.*

In the proof of the proposition we prove that span of control $r_0=(\alpha+1)/(\alpha-1)$ minimizes the function $\xi(r)=(r+1)^{\alpha}/(r-1)$. However, the value r_0 may be non-integer. Therefore, r_* equals to floor integer value $r_-=\lfloor r_0 \rfloor$ (maximal integer is less than or equal to $(\alpha+1)/(\alpha-1)$) or ceil integer value $r_+=\lceil r_0 \rceil$ (minimal integer is greater than or equal to $(\alpha+1)/(\alpha-1)$). Among floor and ceil we choose the value that minimize the function $\xi(r)$. Thus, Proposition 4 leads to the following *optimal span of control*:

$$r_*=\begin{cases} r_-, \text{for } \xi(r_-)<\xi(r_+), \\ r_+, \text{for } \xi(r_+)\le\xi(r_-). \end{cases} \tag{13}$$

Proposition 4 leads to optimal r_*-tree with each manager having exactly r_* immediate subordinates. So, **there exists optimal hierarchy with span of control** r_* if $n-1$ is divisible by r_*-1. For example, for $r_*=3$ the following values of *n* are possible $n=3,5,7,9...$. For *n*=7 the optimal tree is shown in Figure 13. If $n=(r_*)^j$ then we can construct optimal symmetric tree with *j*+1 hierarchical tiers. In this tree any manager on the second tier has exactly r_* immediately subordinated workers. Each manager on the next tier has exactly r_* immediately subordinated managers from the previous tier. For $r_*=3$ and *n*=9 optimal symmetric tree is shown in Figure 14.

If $n-1$ is not divisible by r_*-1 then there does not exist a tree with each manager having exactly r_* immediate subordinates. Proposition 4 implies that for any *n* the cost of an optimal hierarchy is greater than or equal to the following value:

$$(n-1)\lambda^{\alpha}(r_*+1)^{\alpha}/(r_*-1). \tag{14}$$

Moreover, Proposition 3 implies that the cost of an optimal hierarchy is less than or equal to the cost of any decentralized structure controlling the process line. Therefore, **total cost of any managers controlling all flows inside symmetric process line is greater than or equal to (14)**.

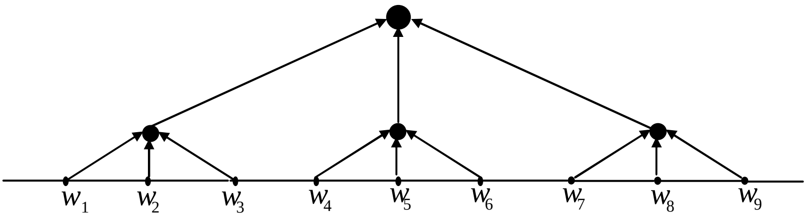

Figure 14. Example of Symmetric Hierarchy over the Process Line

If $n-1$ is divisible by r_*-1 then optimal hierarchy cost reaches its lower bound (14). In this case optimal r_*-tree consists of $q=(n-1)/(r_*-1)$ managers and each of them controls exactly r_* immediate subordinates. If n is arbitrary then the number of managers in optimal tree can be equal to one of two integer numbers closest to $(n-1)/(r_*-1)$. In this case in optimal tree the numbers of managers' immediate subordinates are given by formula (9). For arbitrary n the cost of the optimal tree can not exceed its lower bound (14) more than $1+(r_*-1)/(n-1)$ times[44]. For sufficiently large n the exceeding of the lower bound (14) is insignificant. **Below we suppose that $n-1$ is divisible by r_*-1, so (14) is the optimal hierarchy cost**.

Formula (13) defines the dependence between the optimal span of control and the environment instability α. This dependence is shown in Figure 15. Besides $r_*(\alpha)$ the curve $(\alpha+1)/(\alpha-1)$ is shown in the figure too.

Figure 15 shows that **the optimal span of control decreases when environment instability increases**. This principle is well-known in practice and described in management science literature (see, for instance, Mintzberg (1979)).

[44] We can calculate upper bound of the cost of the tree by choosing the nearest $n_1>n$ such that n_1-1 is divisible by r_*-1. The inequality $n_1-n< r_*-1$ holds. This inequality leads to the estimation mentioned.

If environment stabilizes (α tends to one) then optimal span of control tends to $+\infty$ (one manager controls more and more employees). Particularly, two-tier hierarchy becomes optimal for greater *n* (for firms with greater size). In the limit ($\alpha = 1$) it turns into result for stable environment: two-tier hierarchy with single manager is optimal for any size of the firm. However, **even for small instability ($\alpha > 1$) multi-tier hierarchy is optimal for sufficiently large size of the firm.**

Figure 15 shows that values $\alpha \geq 2.5$ correspond to the *extremely unstable environment*. In this case the optimal span of control equals 2. Thus, in optimal tree each manager has only two immediate subordinates. Total number of managers equals $n-1$. Therefore, optimal tree contains maximal number of managers and each of them controls only one internal flow. So, if the environment is extremely unstable then we have to hire individual manager to control each of the flows.

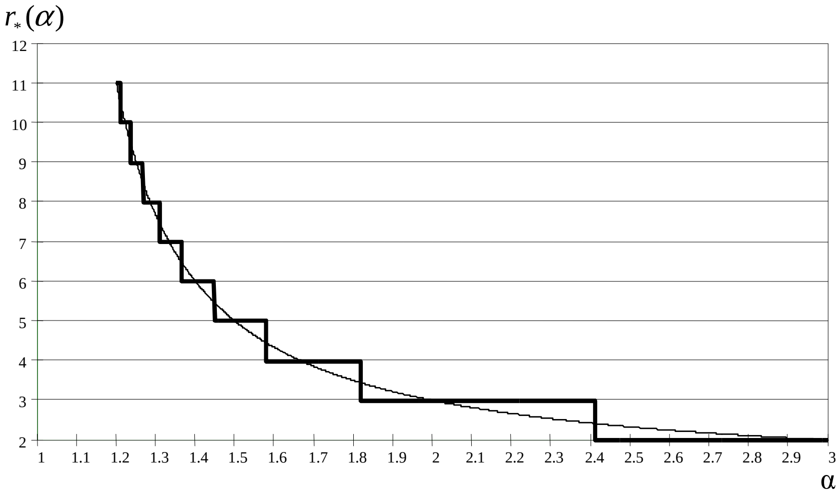

Figure 15. Optimal Span of Control $r_*(\alpha)$ Depending upon Environment Instability α

Usually in real firms each manager has from three to ten immediate subordinates (see Mintzberg (1979)). In some cases the number of immediate subordinates can increase up to hundreds. So, in real firms the hierarchy hits between mentioned above extreme cases (two-tier hierarchy and 2-tree). Therefore, instability range $1 < \alpha < 2.5$ corresponds with most of hierarchies in real firms. In Figure 15 the optimal span of control increases by discrete steps if environment instability α decreases from 2.5 to 1.

If a firm grows (number of workers n increases) then the number of managers and optimal hierarchy cost (14) increase linearly. Therefore, using the model with power cost function and single process line we can not determine limits of the firm growth. Below in Chapter 2 we modify the basic model and obtain optimal hierarchy for several process lines with functional links. This model implies that we must replace the tree (restructure the firm) by more complicated hierarchy if the firm exceeds the limits of growth.

Consider some factors changing intensity λ of the flows controlled by managers. In Section 1.5 we note that labor of managing may consist of operational planning and monitoring of the flows. For such manager's labor Mintzberg (1979) uses the term *direct supervision and authority*. In other words, this is manager's labor, which is necessary to achieve operational goals of the firm.

Usually managers do not need control (directly supervise) all flows in the technological network. Some part of the flows does not require the control because workers can control it themselves. Mintzberg (1979) argues that in real firm *standardization* increases the part of the flow, which does not require managers' control. Thus, standardization decreases the cost of direct supervision and authority. Mintzberg considers some *types of standardization*:

1. Standardization of skills and knowledge provides workers' self-dependent coordination because the workers have some knowledge and habits of work in standard situations.

2. Standardization of output defines requirements for each worker's product. Due to the standardization of output workers can solve routine problems with low-quality products with no participation of the managers.

3. Standardization of work processes defines the job descriptions. These descriptions regulate worker's activity and decrease manager's participation in the work processes.

Therefore, all types of standardization decrease the part of the flow, which requires managers' control.

In basic model standardization can be taken into consideration using appropriate cost function. For example, let's define standardization index $0 \le s \le 1$. In this case we can consider cost function $\varphi(x) = (x(1-s))^{\alpha}$ instead of the function $\varphi(x) = x^{\alpha}$. If standardization index equals to zero then all flows must be controlled. If standardization is complete (s=1) then direct supervision and authority become unnecessary. But it is more convenient to suppose that standardization does not change the cost function but changes the intensity of the flows (in the example all flows are multiplied by 1–s). It does not change optimal hierarchy problem from mathematical point of view, but the results can be interpreted easily.

Thus, we suppose that **for symmetric process line flow intensity λ is maximal with no standardization and equals to zero with the complete standardization**. So, λ is the intensity of flows, which require managers' control (direct supervision and authority).

In some real firms increase of standardization does not change the span of control but decreases managers' costs (Mintzberg (1979)). Basic model with power cost function and symmetric process line leads to similar results. Increase of standardization decreases λ and managers' costs (see expression (14)). But increase of standardization does not change the optimal span of control r_* (see expression (13)). So, basic model explains these practical effects.

On the whole basic model defines terminology and explains optimal hierarchy problem considered in this paper. The results obtained for symmetric process line are used in Chapter 2 to solve optimal hierarchy problem for several process lines with functional links (for this problem we prove optimality of divisional, functional or matrix hierarchy). Basic model and Chapter 2 show that using some examples of sectional cost functions[45] depending on flows we can model many practical effects observed in a real firms. Thus, it is important to analyze the whole class of sectional functions. In Chapter 3 we explore general model (arbitrary sectional cost functions).

[45] Let's remind that sectional cost function depends only on groups controlled by immediate subordinates of a manager (see the formal definition on page 79).

2. Functional, Divisional and Matrix Hierarchy Optimality

Advantages and disadvantages of divisional, functional and matrix hierarchies depend first of all on nature of workers' interactions (see management science literature, for instance, Mintzberg (1979)). For example, these advantages and disadvantages may depend on flows of technological network. Therefore, in Chapter 2 we consider technological network composed of several process lines with functional links between workers in different lines. For some cost functions such type of the network allows to prove optimality of typical hierarchy (divisional, functional or matrix) for any size of the firm. Moreover, using these cost functions and network we can model many empirical dependences well known in practice. For example, we model dependences between type of optimal hierarchy and environment instability, standardization, intensity of product and functional flows, horizontal and vertical integration, etc.

Sections 2.1 and 2.2 describe flows of technological network composed of process lines with functional links. In Section 2.3 all managers are divided into several types. Division, department and typical hierarchies (divisional, functional and matrix) are defined formally. Sections 2.4-2.6 describe type and properties of considered cost function. For this function in Section 2.7 we prove optimality of a typical hierarchy. Section 2.8 compares different typical hierarchies and analyses dependences between type of optimal hierarchy and different parameters of the model.

2.1. Process Lines with Functional Links. Product and Functional Flows

In Section 1.11 for power cost function we obtain optimal hierarchy controlling symmetric process line. Below we use this result to obtain optimal hierarchy controlling more complex technological network with several process lines and functional links. We describe mathematical model, which allows to define and compare divisional, functional and matrix hierarchy controlling this technological network (see Figure 16).

Suppose the technological network is composed of *l* process lines ($l \geq 2$). Each process line produces some product (or provides services for some region, customers, etc.). The production requires some technological operations. Suppose *n* workers carry out these operations. Each process line contains *n* workers ($n \geq 2$).

So, in the technological network the set of workers is given by $N=\{w_{i,j}\}$, where $1 \leq i \leq l$, $1 \leq j \leq n$. The index *i* is the number of the process line, which contains the worker. The index *j* is the number of the worker in the line (or the number of the operation performed by the worker). Thus, the set *N* consists of *nl* workers. Each worker is defined by two inferior indexes *i*, *j*.

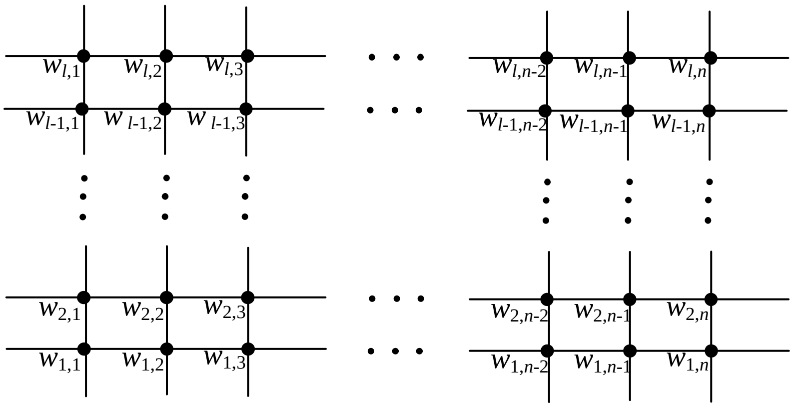

Figure 16. Process Lines with Functional Links
(Network with Product and Functional Flows)

Consider the process line with number *i*. Worker $w_{i,1}$ may, for example, purchase raw materials. $w_{i,1}$ passes raw materials to the next worker $w_{i,2}$ in the line. Worker $w_{i,2}$ executes some technological operation and passes its results to the next worker $w_{i,3}$, etc. The last worker $w_{i,n}$ in the process line may, for example, dispatch finished product to the customers. So, we suppose that there are *product flows* between the worker $w_{i,j}$ and nearest-neighbor workers $w_{i,j\text{-}1}$ and $w_{i,j+1}$ in the process line. These flows allow to produce finished product of the process line. There is product flow between the first worker in the line and the environment (for example, purchasing raw materials) in addition to the flow between the first and the second workers. Similarly there is product flow between the last worker in the line and the environment (for example, dispatching finished product). Suppose intensities of all product flows are the same. Thus, we consider symmetric process lines with the same flow intensity in different lines.

Suppose workers with the same number perform similar technological operations in different process lines. So, workers with the same number have similar professional skills, may use common equipment, etc. For example, in all process lines first workers $w_{1,1}$, $w_{2,1}, \ldots, w_{l-1,1}$, $w_{l,1}$ are responsible for purchasing raw materials. Therefore, these workers must have skills that allow to interact with vendors, compare different business offers, choose the best vendors, etc. Thus, workers with the same number interact with each other and there are flows (informational, material, etc.) between these workers. For example, a worker $w_{i,1}$ is able to gather from another first workers information about price change, about new type of raw materials on the market, about the best vendors, etc. And we suppose that a worker may gather such information only from workers in "nearest-neighbor" lines. For example, process lines may be located as it is shown in Figure 16. In this case "nearest-neighbor" lines are lines with previous and next numbers. So, we suppose that there are *functional flows* between the worker $w_{i,j}$ and workers $w_{i-1,j}$ and $w_{i+1,j}$ in nearest-neighbor process lines. There is functional flow between the worker in the first line and the environment (for example, professional contacts with experts in other firms) in addition to the flow with the worker in the second line. Similarly there is functional flow between the worker in the last line (with number l) and the environment. Suppose intensities of all functional flows are the same. Thus, we consider symmetric "*functional lines*" with the same flow intensity in different lines. Functional lines are numbered from 1 to n (the first line may correspond with purchasing, the last line may correspond with production distribution).

So, the technological network in Figure 16 consists of *process lines with functional links*. Lines with product and functional flows are crossed. Each worker belongs to one process line and one functional line. For example, a process line may contain material flows and a functional line may contain informational flows, essential for providing material flows and producing finished products.

Let $N_i=\{w_{i,1},\ldots,w_{i,n}\}$ be the process line with the number i. Union of process lines $N_1,\ldots,N_l$ is the whole set of workers: $N = N_1 \cup \ldots \cup N_l$.

Let $N^j=\{w_{1,j},\ldots,w_{l,j}\}$ be the functional line with the number j. Union of functional lines $N^1,\ldots,N^n$ is the whole set of workers: $N = N^1 \cup \ldots \cup N^n$.

In the following section we define intensity of flows in the technological network.

2.2. Product and Functional Flows Intensity

Just as in Chapter 1 (basic model) in this chapter we suppose that managers' cost depends on flows of the technological network. In Section 1.11 we note that manager's labor may correspond with "direct supervision" of the flows. Standardization decreases the intensity of flows, which require managers' control (direct supervision).

Let $\lambda > 0$ be the intensity of such part of the product flow that must be controlled by managers.

Let $\theta > 0$ be the intensity of such part of the functional flow that must be controlled by managers.

Suppose **standardization decreases both part of product flow and part of functional flow that must be controlled**. Without standardization all flows must be controlled. In this case λ and θ are maximal values, that correspond with all flows in the technological network. If standardization is close to complete, then intensities λ and θ are close to zero.

Usually intensities λ and θ are related. If product flow (production volume) increases, then functional flow (functional interaction) may increase too. It corresponds with practice, when production volume increase causes increase of cost of managers controlling both product and functional flows. We suppose nothing about relation between intensities of product and functional flows. Instead we explore the model for any λ and θ.

Generally both product and functional flows may consist of several components (for example, different types of material flows). Thus,

λ and θ may be some vectors. However, below in Chapter 2 we suppose that **any flow has single-component**. So, λ and θ are some positive real numbers.

Therefore, the flow function $f(\cdot)$ is given by the following expressions. For each $1 \le i \le l$ and $1 \le j \le n$ the worker $w_{i,j}$ has four links:

$$f(w_{i,j-1}, w_{i,j}) = f(w_{i,j}, w_{i,j+1}) = \lambda,$$
$$f(w_{i-1,j}, w_{i,j}) = f(w_{i,j}, w_{i+1,j}) = \theta. \qquad (15)$$

If in expressions (15) index j–1 equals to zero (or index j+1 is greater than n), then we mean that the "worker" $w_{i,0}$=$w_{i,n+1}$= w_{env}^{prod} is part of the environment linked with workers by product flows. w_{env}^{prod} will be called *product environment*. Similarly we mean that the "worker" $w_{0,j}$=$w_{l+1,j}$= w_{env}^{func} is part of the environment linked with workers by functional flows. w_{env}^{func} will be called *functional environment*. All flows are defined by expressions (15). There are no other flows in the technological network.

Expressions (15) imply that each process line contains product flows with intensity λ. Each functional line contains functional flows with intensity θ. For the first process line and the first functional line intensities are shown in Figure 17. For other lines intensities are similar.

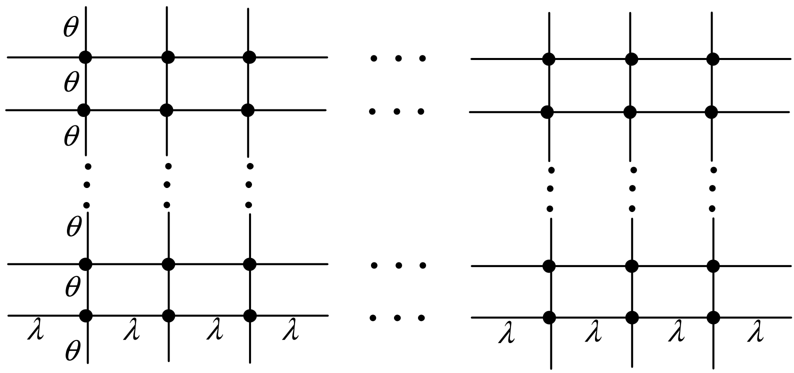

Figure 17. Intensities of Product and Functional Flows in the First Process Line and the First Functional Line

The technological network (see Figure 16) and special flow intensities (see Figure 17) strongly restrict technology. Particularly, all process lines must contain the same number of workers *n*. Workers with the same number must perform similar operations and have functional links. Real firm may differ from this model. For example, one worker may purchase raw materials for all process lines; lengths of process lines may differ; some lines may contain workers performing unique operations, not essential in other lines, etc. Flow intensity may change for different lines or different parts of a line. However, in some cases technological network of real firm may be approximately modeled using process lines with functional links. Such type of the network allows to explain optimal hierarchy problem analytically. If technological network is much more complex then to obtain optimal hierarchy we can use methods described in Chapter 3 (general model).

Considered technological network (see Figures 16 and 17) allows to define divisional, functional and matrix hierarchies formally (see the following section).

2.3. Divisions and Departments. Typical Hierarchies

Each manager of a hierarchy controls some group in the technological network. Using this fact we define several types of managers.

Divisional manager is a manager controlling only workers in one process line. If a divisional manager *m* controls all workers in the process line then manager *m* will be called *division head*. Manager *m* and all subordinated employees will be called *division*.

Suppose the number of the division equals to number of the process line controlled by this division. If in the hierarchy there is no manager controlling all process line then there is no division with corresponding number (even if different divisional managers control all process line by parts).

If *m* is divisional manager, then $s(m) \subseteq N_i$, where $1 \le i \le l$ is the number of the process line, $s(m)$ is the group of workers controlled by *m*. If some manager m_1 is subordinated to the manager *m*, then $s(m_1) \subset s(m)$ (see Lemma 1). So, a divisional manager may control only divisional managers or workers.

Similarly we can define managers controlling functional line.

Functional manager is a manager controlling only workers in one functional line. If a functional manager m controls all workers in the functional line, then manager m will be called *head of department*. Manager m and all subordinated employees will be called *department*.

Suppose the number of the department equals to number of the functional line controlled by this department. If in the hierarchy there is no manager controlling all functional line, then there is no department with corresponding number (even if different functional managers control all functional line by parts). A functional manager may control only functional managers or workers.

Divisional and functional managers will be called *middle-tier managers*. Moreover, let's define two types of *strategic managers*:

1. *Manager controls interactions between divisions* if each immediately subordinated manager controls one process line (this manager is the division head) or several process lines.

2. *Manager controls interactions between departments* if each immediately subordinated manager controls one functional line (this manager is the head of the department) or several functional lines.

Consider process lines with functional links (the network N).

Divisional hierarchy is a hierarchy from $\Omega(N)$ consisting of l divisions and strategic managers controlling interactions between divisions.

Functional hierarchy is a hierarchy from $\Omega(N)$ consisting of n departments and strategic managers controlling interactions between departments.

Matrix hierarchy is a hierarchy from $\Omega(N)$ consisting of l divisions, n departments and the top manager immediately controlling all division heads and heads of all departments.

Consider an example with $l=3$ and $n=9$. Thus, there are three process lines with nine workers in each line (with nine functional lines).

An example of divisional 3-hierarchy is shown in Figure 18. The hierarchy contains three divisions. Each division controls one process line (controls all flows for one finished product). Each division consists of the division head, three immediately subordinated managers and workers of the process line. Divisional hierarchy in Figure 18 contains single strategic manager. He or she controls interactions between three divisions.

An example of functional 3-hierarchy is shown in Figure 19. The hierarchy contains nine departments. Each department controls one functional line (controls all flows for one kind of activity, i.e. flows between workers with the same number). Each department consists of the head of the department and workers of the functional line. Functional hierarchy in Figure 19 contains four strategic managers. The first strategic manager controls interactions between departments 1, 2, 3, the second one – between departments 4, 5, 6, the third one – between departments 7, 8, 9. These three managers are immediately subordinated to the top manager. This manager controls other interactions between departments.

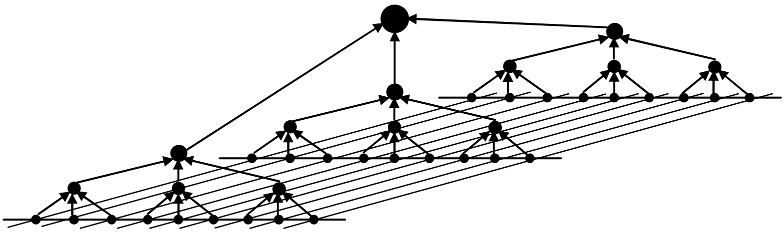

Figure 18. An Example of Divisional 3-Hierarchy

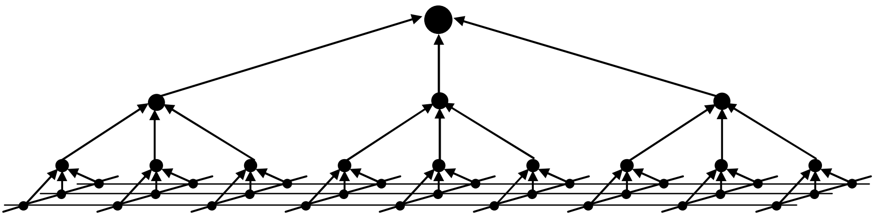

Figure 19. An Example of Functional 3-Hierarchy

An example of matrix hierarchy is shown in Figure 20. The hierarchy contains three divisions. Each division controls one process line (similarly with the divisional hierarchy in Figure 18). Moreover, the matrix hierarchy contains nine departments. Each department controls one functional line (similarly with the functional hierarchy in Figure 19). Dotted lines correspond with functional links and subordination edges in departments. The definition implies that the matrix hierarchy contains single top manager immediately controlling all division heads and heads of all departments. To simplify Figure 20 we do not draw the top manager.

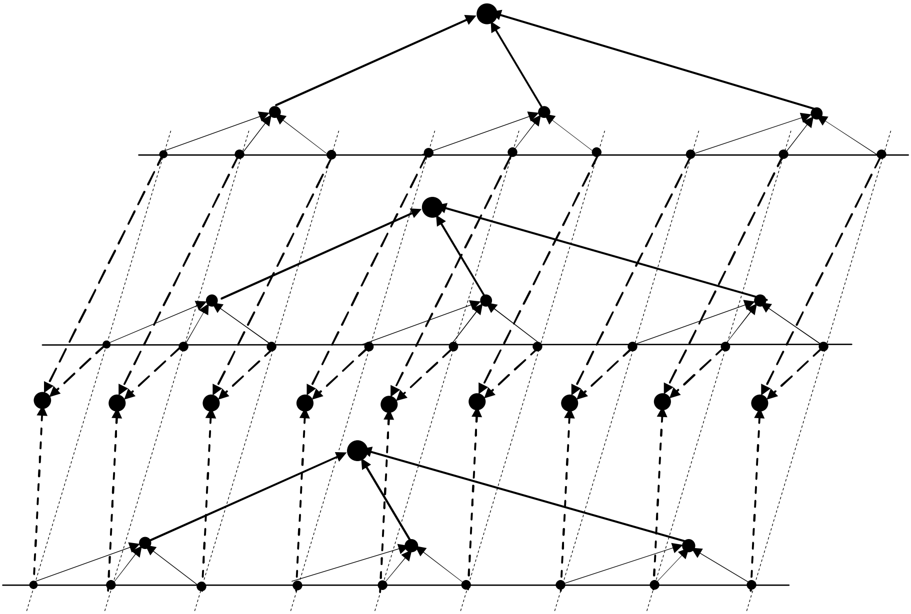

Figure 20. An Example of Matrix Hierarchy
(the top manager is not drawn)

Divisional, functional and matrix hierarchies will be called *typical hierarchies* because such hierarchies are often used in real firms (Mintzberg (1979)). To compare different hierarchies it is necessary to define managers' costs. In Sections 2.4, 2.5 and 2.6 we define cost function for different types of managers and discuss interpretations.

2.4. Fixed and Variable Cost

In Section 1.11 for power cost function we find optimal hierarchy controlling single process line[46]. Below we use these results. To define

[46] I.e. the line, which is not linked with other lines in technological network.

fixed and variable cost let's consider a single symmetric line with flow intensity λ.

Let k be the number of manager's flows. All flows have the same intensity λ. So, the power cost function defines manager's cost in the following way $\varphi(k\lambda) = (k\lambda)^\alpha$, where α is exponent (environment instability, see expression (10) on page 39).

Thus, in Section 1.11 we consider manager's cost depending only on total intensity of the controlled flows. If the intensity equals to zero then manager's cost equals to zero too. However, in practice there is some non-zero fixed cost for each controlling link even if the flow intensity equals to zero for this link. Therefore, in contrast to Section 1.11 below in this chapter we consider more realistic cost function with fixed and variable cost for each link controlled by the manager. Let's define formally variable and fixed cost.

So, consider a manager m who controls and participates in control of k flows with the same intensity λ. Then *variable cost* $(k\lambda)^\alpha$ *of the manager m depends on total intensity of the manager's flows*.

Suppose *in stable environment there exists some fixed controlling cost* $c_0 > 0$ *for each link*. For example, the manager can periodically make reports about actual flow for the link. Such cost does not depend on flow intensity because such manager's efforts do not depend on intensity value in reports. Suppose the constant c_0 is the same both for product and for functional flows. So, fixed cost depends on neither intensity, nor type of flow.

For the manager m there is fixed cost for each of k flows. Thus, in stable environment fixed cost of the manager equals kc_0. This cost may depend on environment instability. For example, instability may change forms of reports. The manager must adapt to new forms. The more instability, the more both variable and fixed cost. Therefore, we suppose that *fixed cost of the manager m equals* $(kc_0)^\alpha$.

So, in this chapter we consider the following *manager's cost function*:

$$(k\lambda)^\alpha + (kc_0)^\alpha = k^\alpha(\lambda^\alpha + c_0^\alpha), \tag{16}$$

where k is the number of manager's flows, λ is the intensity of each flow, c_0 is fixed cost for each flow in stable environment, $\alpha > 1$ is environment instability.

In the basic model (Chapter 1) we consider manager's cost function $\varphi(F^{\text{int}}(m)+F^{ext}(m))$. For symmetric line with the intensity λ this function is given by $\varphi((k_1+k_2)\lambda)$, where k_1 and k_2 are numbers of internal and external flows of the manager m. For power function $\varphi(\cdot)$ and symmetric line the cost of the manager equals $k^{\alpha}\lambda^{\alpha}$, where $k = k_1+k_2$ is the number of manager's flows. In expression (16) the cost of the manager equals $k^{\alpha}(\lambda^{\alpha}+c_0^{\alpha})$. So, we only change the multiplier λ^{α} to the multiplier $(\lambda^{\alpha}+c_0^{\alpha})$ for all managers of any hierarchy. Also for the cost of any hierarchy the multiplier λ^{α} are changed to the multiplier $(\lambda^{\alpha}+c_0^{\alpha})$. Therefore, **for the function (16) with variable and fixed cost the optimal hierarchy controlling single symmetric line is the same as for the power cost function (10)**. Thus, all propositions introduced in Section 1.11 hold for cost function (16). Let's briefly describe several facts that are used below.

Optimal span of control r_* is given by expression (13) (see page 42). In expression (14) the multiplier λ^{α} is changed to the multiplier $(\lambda^{\alpha}+c_0^{\alpha})$. So, **total cost of any managers controlling all flows inside symmetric line is greater than or equal to the following value**:

$$(n-1)(\lambda^{\alpha}+c_0^{\alpha})(r_*+1)^{\alpha}/(r_*-1). \tag{17}$$

If $n-1$ contains r_*-1, then for single symmetric line r_*-tree with each manager immediately controlling r_* subordinates is optimal hierarchy. The cost of this tree is given by expression (17). Below we consider process and functional lines. And we suppose that both $n-1$ and $l-1$ contains r_*-1. Thus, **both for single process line and for single functional line r_*-tree is optimal hierarchy**.

2.5. Middle-Tier and Strategic Managers' Flows and Cost

In Section 2.3 we define two types of middle-tier managers and two types of strategic managers. In this section we define flow and cost for each type of manager.

Cost of Divisional Manager and Optimal Division

Any divisional manager controls only workers in one process line. Let m be some divisional manager controlling workers in the process line with number i ($s(m) \subseteq N_i$). There are only product flows inside the group $s(m)$ controlled by the manager. So, *a divisional manager controls only internal product flows*. There are product flows between workers in the group $s(m)$ and other workers in N_i or product environment w_{env}^{prod} . Thus, *a divisional manager participates in control of external product flows of the group* $s(m)$. Moreover, there are functional flows between workers in the group $s(m)$ and other process lines or functional environment w_{env}^{func} . However, the divisional manager does not participate in control of functional interactions because he or she is responsible only for the product of the process line. So, we suppose that *a divisional manager does not participate in control of functional flows*.

For example, a division is shown in Figure 21. It is part of the divisional hierarchy (see Figure 18). Manager m immediately controls three managers: m_1, m_2 and m_3. m controls two internal product flows and participates in control of two external product flows (thick lines in Figure 21). However, m does not participate in control of external functional flows (dotted lines in Figure 21).

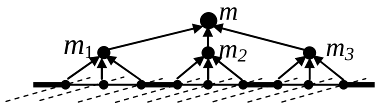

Figure 21. An Example of Division

So, the cost of a divisional manager equals to the cost of corresponding manager controlling single process line with flow intensity λ. Therefore, expression (16) implies that **the cost of the divisional manager is given by**:

$$(k\lambda)^{\alpha} + (kc_0)^{\alpha} = k^{\alpha}(\lambda^{\alpha} + c_0^{\alpha}), \tag{18}$$

where k is the number of manager's product flows.

Total cost of the whole division equals to the cost of the corresponding hierarchy controlling single process line. Therefore, r_*-tree with each manager immediately controlling r_* subordinates is the division with minimal cost. This tree will be called *optimal division*. The division controls n workers and product flows with intensity λ (see

Figures 16 and 17). So, expression (17) implies that **the cost of the optimal division is given by**:

$$(n-1)(\lambda^{\alpha}+c_0^{\alpha})(r_*+1)^{\alpha}/(r_*-1)\,. \tag{19}$$

This is the minimal possible total cost of any managers controlling all product flows of the process line with the length *n*.

Cost of Functional Manager and Optimal Department

Let's define cost of a functional manager.

In the firm a functional manager is responsible for one type of activity. This manager controls workers in one functional line. These workers perform similar operations. Let *m* be some functional manager controlling workers in the functional line with number j ($s(m)\subseteq N^j$). There are only functional flows inside the group *s*(*m*) controlled by the manager. So, *a functional manager controls only internal functional flows*. There are functional flows between workers in the group *s*(*m*) and other workers in N^j or functional environment w_{env}^{func}. Thus, *a functional manager participates in control of external functional flows of the group s*(*m*). Moreover, there are product flows between workers in the group *s*(*m*) and other functional lines or product environment w_{env}^{prod}. However, the functional manager does not participate in control of product interactions because he or she is responsible only for control of interactions between workers performing similar operations inside the functional line. So, we suppose that *a functional manager does not participate in control of product flows*. The cost of the functional manager may be related with production volume indirectly. If the intensity of product flows changes, then the intensity of functional flows may change too. But for fixed λ and θ we suppose that the cost of the functional manager does not depend on product flows intensity.

For example, a department is shown in Figure 22. It is part of the functional hierarchy (see Figure 19). Manager *m* immediately controls three workers: $w_{1,j}$, $w_{2,j}$ and $w_{3,j}$. *m* controls two internal functional flows and participates in control of two external functional flows (solid lines in Figure 22). However, *m* does not participate in control of external product flows (dotted lines in Figure 22).

So, the cost of a functional manager equals to the cost of corresponding manager controlling single functional line with flow intensity

θ. Therefore, expression (16) implies that **the cost of the functional manager is given by**:

$$(k\theta)^{\alpha} + (kc_0)^{\alpha} = k^{\alpha}(\theta^{\alpha} + c_0^{\alpha}), \tag{20}$$

where k is the number of manager's functional flows.

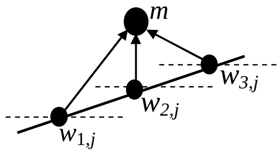

Figure 22. An Example of Department

Total cost of the whole department equals to the cost of the corresponding hierarchy controlling single functional line. Therefore, r_*-tree with each manager immediately controlling r_* subordinates is the department with minimal cost. This tree will be called *optimal department*. The department controls l workers and functional flows with intensity θ (see Figures 16 and 17). So, expression (17) implies that **the cost of the optimal department is given by**:

$$(l-1)(\theta^{\alpha} + c_0^{\alpha})(r_* + 1)^{\alpha}/(r_* - 1). \tag{21}$$

This is the minimal possible total cost of any managers controlling all functional flows of the functional line with the length l.

Cost of Strategic Manager Controlling Interactions between Divisions

Let m be some strategic manager controlling interactions between divisions. Any his or her immediately subordinated manager controls one division or several divisions. For example, in Figure 18 the top manager immediately control three managers. Each of them controls one division. Manager m controls several divisions. In Figure 18 first, second and third division heads are immediately subordinated to m. Generally each of immediately subordinated managers may control several divisions.

Let's define the flows of the manager m. His or her subordinates control all product flows inside subordinated process lines. Therefore, *m controls only functional flows inside subordinated group* $s(m)$. So, m controls only functional interactions between process lines (for example, in Figure 18 the top manager *m* controls eighteen internal functional flows). Also *manager m participates in control of external functional flows of the group* $s(m)$ (in Figure 18 there are eighteen functional flows between divisions and the functional environment). Moreover, there are product flows between each division and the product environment. Immediate subordinates of the strategic manager or their subordinated managers participate in these flows control. Suppose these managers are entirely responsible for output of products. So, we suppose that manager *m does not participate in control of product flows*. Thus, subordinates "hide" the problems of output of each specific product from the strategic manager.

Intensity of total functional flow between two nearest-neighbor divisions equals $n\theta$. Therefore, expression (16) implies that the cost of the strategic manager controlling interactions between divisions is given by:

$$(kn\theta)^{\alpha} + (kc_0)^{\alpha} = k^{\alpha}((n\theta)^{\alpha} + c_0^{\alpha}), \qquad (22)$$

where *k* is the number of such functional interactions between divisions that the manager controls these interactions or participates in control.

Suppose there are *l* divisions in the hierarchy and strategic managers control all functional interactions between these divisions. If a strategic manager immediately controls a divisional manager, then this manager is division head. Therefore, the strategic managers actually control "line" with *l* division heads and functional flows with intensity $n\theta$. For example, in Figure 18 the top manager controls "line" with 3 division heads and functional flows with intensity 9θ. We can substitute corresponding values in expression (17) and obtain *the lower bound of total cost of strategic managers controlling functional interactions between l divisions*:

$$(l-1)((n\theta)^{\alpha} + c_0^{\alpha})(r_* + 1)^{\alpha} / (r_* - 1). \qquad (23)$$

Moreover we can construct r_*-tree with cost (23), that controls division heads instead of workers and consists of strategic managers (each of them has r_* immediate subordinates). So, **r_*-tree is the hierarchy, that consists of strategic managers and with minimal cost controls functional interactions between *l* divisions**.

Cost of Strategic Manager Controlling Interactions between Departments

Similarly we can define cost of strategic manager controlling interactions between departments. Let *m* be such strategic manager. Any his or her immediately subordinated manager controls one department or several departments. For example, in Figure 19 the top manager immediately controls three managers. Each of them controls three departments.

Let's define the flows of the manager *m*. His or her subordinates control all functional flows inside subordinated functional lines. Therefore, *m controls only product flows inside subordinated group s*(*m*). So, *m* controls only product interactions between functional lines (for example, in Figure 19 the top manager *m* controls six internal product flows). Also *manager m participates in control of external product flows of the group s*(*m*) (in Figure 19 there are six product flows between departments and the product environment). Moreover, there are functional flows between each department and the functional environment. Immediate subordinates of the strategic manager or their subordinated managers participate in these flows control. Suppose these managers are entirely responsible for functional interactions. So, we suppose that manager *m does not participate in control of functional flows*. Thus, subordinates "hide" the problems of each specific operation from the strategic manager. For example, the strategic manager can define production plans for subordinated departments and monitor only the fact of plans realization but not the realization process.

Intensity of total product flow between two nearest-neighbor departments equals $l\lambda$. Therefore, expression (16) implies that the cost of the strategic manager controlling interactions between departments is given by:

$$(kl\lambda)^{\alpha} + (kc_0)^{\alpha} = k^{\alpha}((l\lambda)^{\alpha} + c_0^{\alpha}), \tag{24}$$

where *k* is the number of such product interactions between departments that the manager controls these interactions or participates in control.

Suppose there are *n* departments in the hierarchy and strategic managers control all product interactions between these departments. If a strategic manager immediately controls a functional manager then this manager is the head of the department. Therefore, the strategic managers actually control "line" with *n* heads of departments and product flows with intensity $l\lambda$. For example, in Figure 19 strategic managers control "line" with 9 heads of departments and product flows with intensity 3λ. We can substitute corresponding values in expression (17) and obtain *the lower bound of total cost of strategic managers controlling product interactions between n departments*:

$$(n-1)((l\lambda)^{\alpha}+c_0^{\alpha})(r_*+1)^{\alpha}/(r_*-1)\ . \tag{25}$$

Moreover we can construct r_*-tree with cost (25), that controls heads of departments instead of workers and consists of strategic managers (each of them has r_* immediate subordinates). So, ***r**-tree is the hierarchy, that consists of strategic managers and with minimal cost controls product interactions between *n* departments**.

2.6. Cost Function

Expressions (18) and (20) define costs of middle-tier managers. Expressions (22) and (24) define costs of strategic managers. Using these expressions we can define the cost function for any manager in a hierarchy.

For arbitrary hierarchy $H \in \Omega(N)$ controlling process lines with functional links the cost of the manager m is given by the following function:

$$c(s_H(v_1),\ldots,s_H(v_k)) =$$

$$= \begin{cases} k_1^{\alpha}(\lambda^{\alpha}+c_0^{\alpha}) \text{ for divisional manager;} \\ k_2^{\alpha}(\theta^{\alpha}+c_0^{\alpha}) \text{ for functional manager;} \\ k_3^{\alpha}((l\lambda)^{\alpha}+c_0^{\alpha}) \text{ for strategic manager} \\ \quad \text{controlling interactions between departments;} \\ k_4^{\alpha}((n\theta)^{\alpha}+c_0^{\alpha}) \text{ for strategic manager} \\ \quad \text{controlling interactions between divisions;} \\ 0, \text{ for other managers with } F_H^{\text{int}}(m)=0; \\ +\infty, \text{ for other managers with } F_H^{\text{int}}(m)>0, \end{cases} \tag{26}$$

where $v_1,\ldots,v_k$ are immediate subordinates of manager m, k_1 is the number of flows of the divisional manager, k_2 is the number of flows of the functional manager, k_3 is the number of such product interactions between departments that the manager controls these interactions or participates in control, k_4 is the number of such functional interactions between divisions that the manager controls these interactions or participates in control.

Expression (26) implies that the cost of the manager depends only on groups $s_H(v_1),\ldots,s_H(v_k)$ controlled by immediate subordinates of the manager. Indeed, group $s_H(m) = s_H(v_1) \cup \ldots \cup s_H(v_k)$ controlled by m depends only on these groups. If $s_H(m)$ is embedded into a process or a functional line then m is divisional or functional manager correspondingly. If each of groups $s_H(v_1),\ldots,s_H(v_k)$ consists of process lines or functional lines then m is strategic manager with corresponding type. Also values k_1, k_2, k_3, k_4 and manager's internal flow $F_H^{\text{int}}(m)$ (see Lemma 3 on page 20) depend only on groups $s_H(v_1),\ldots,s_H(v_k)$ and the groups $s_H(m)$.

Thus, cost function (26) is given in the form of sectional function[47] $c(s_H(v_1),\ldots,s_H(v_k))$ just as the cost of the manager in the basic model (see expression (4) on page 21). The basic model and the model described in this chapter illustrates that sectional functions can be useful for modeling optimal hierarchies in firms. The class of all sectional functions is explored in Chapter 3. Below in this chapter we solve optimal hierarchy problem for function (26).

For cost function (26) we can construct a hierarchy containing managers that do not control any internal flow ($F_H^{\text{int}}(m) = 0$). The cost of such manager equals to zero. For example, in the matrix hierarchy the top manager immediately controls all division heads and all heads of departments. All flows inside the technological network are controlled by subordinates of the top manager. Therefore, we suppose that *the top manager of the matrix hierarchy does not participate in control of any flow*. The top manager performs some other functions (for example, makes decisions in the case of conflict between division heads and heads of departments). However, in the basic model and in this chapter the manager's cost depends only on flows of the technological network. Thus, we suppose that in the matrix hierarchy the cost of the top manager equals to zero because he or she does not control flows. Similarly we suppose that the cost of any manager equals to zero if this manager does not control internal flows.

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[47] See the formal definition on page 79.

Cost function (26) prohibits the hierarchies with some manager controlling both product and functional flows (the cost of such manager equals to infinity). So, we suppose that the cost of a "universal" (non-specialized) manager is too high because the manager performs too diversified functions. Such manager is shown in Figure 23.

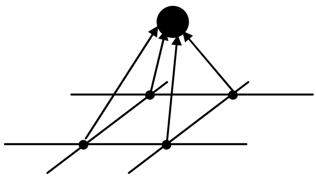

Figure 23. The Manager Controls both Product and Functional Flows

Using function (26) we also suppose that **only strategic manager can control interactions between divisions or departments**. It is necessary that skilled subordinated managers help a strategic manager. For example, consider some strategic manager controlling interactions between divisions. Division heads and subordinated managers control all product flows inside the division. And the strategic manager only controls functional interactions between divisions. If only parts of divisions are organized (there are no division heads) then the cost of a manager controlling these parts is too high because he or she must participate in control of both product and functional flows.

So, cost function (26) prohibits the hierarchies with some manager controlling interactions between several parts of different departments or divisions (the cost of such manager equals to infinity). For example, in Figure 24 managers m_1 and m_2 control interactions between parts of divisions instead of the whole divisions. Therefore, managers m_1 and m_2 are neither strategic managers nor middle-tier managers. Thus, function (26) implies that their costs equal to infinity.

Consider cost function (26) and some optimal hierarchy controlling process lines with functional links. Any product flow inside a process line is controlled by some manager m. So, this flow is internal for manager m. Therefore, $F^{\text{int}}(m) > 0$. The cost of manager m is finite because he or she is a member of the optimal hierarchy. Thus, function (26) implies that m is either divisional manager or strategic manager controlling interactions between departments. Similarly we can consider a functional flow.

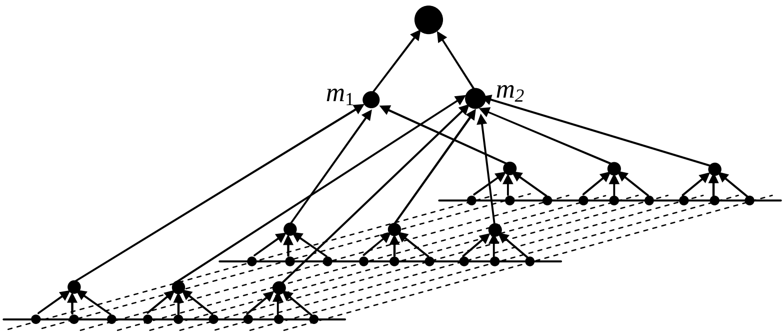

Figure 24. Managers m_1 and m_2 Control Interactions between Parts of Divisions

As a result, we obtain the following statements. For cost function (26) in an optimal hierarchy any product flow is controlled by a divisional manager or a strategic manager controlling interactions between departments. Any functional flow is controlled by a functional manager or a strategic manager controlling interactions between divisions. Therefore, for cost function (26) **we can consider only hierarchies with all flows controlled by middle-tier managers or strategic managers**. Using this fact we obtain the optimal hierarchy in the next section.

2.7. Typical Hierarchies Optimality

Let's consider a divisional hierarchy (see the example in Figure 18) with minimal cost among all divisional hierarchies. A strategic manager can immediately control division heads and can not immediately control other divisional managers. Therefore, any division can be reconstructed independently from other divisions and strategic managers (i.e. costs of other divisions and strategic managers do not change). If there is non-optimal division in the hierarchy then we can reconstruct this division and decrease the cost of the whole hierarchy. Thus, the divisional hierarchy with minimal cost contains only optimal divisions.

The cost of an optimal division is given by expression (19). Minimal total cost of strategic managers controlling functional interactions between *l* divisions is given by expression (23). So, **the divisional *r**-hierarchy** $H_{divisional}$ **has minimal cost**. In this hierarchy each manager controls r_* immediate subordinates. The cost of this hierarchy is given by:

$$c(H_{divisional}) = \frac{(r_* + 1)^\alpha}{r_* - 1}[l(n-1)(\lambda^\alpha + c_0^\alpha) + (l-1)((n\theta)^\alpha + c_0^\alpha)], \quad (27)$$

where r_* is optimal span of control depending on environment instability (see expression (13) on page 42). For example, if $\alpha = 2$ then the divisional 3-hierarchy (see Figure 18) has minimal cost. In expression (27) the first item corresponds with total cost of *l* divisions. The second item corresponds with total cost of strategic managers controlling interactions between divisions. The common multiplier is taken out of the brackets.

We can repeat similar reasoning for functional hierarchy (see the example in Figure 19) with minimal cost among all functional hierarchies. Therefore, the functional hierarchy with minimal cost contains only optimal departments. The cost of an optimal department is given by expression (21). Minimal total cost of strategic managers controlling product interactions between *n* departments is given by expression (25). So, **the functional *r**-hierarchy** $H_{functional}$ **has minimal cost**. The cost of this hierarchy is given by:

$$c(H_{functional}) = \frac{(r_* + 1)^\alpha}{r_* - 1}[n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha)]. \quad (28)$$

For example, if $\alpha = 2$ then the functional 3-hierarchy (see Figure 19) has minimal cost. In expression (28) the first item corresponds with total cost of *n* departments. The second item corresponds with total cost of strategic managers controlling interactions between departments.

Let's consider a matrix hierarchy (see the example in Figure 20) with minimal cost among all matrix hierarchies. The hierarchy consists of *l* divisions, *n* departments and the top manager who immediately controls division heads and heads of departments. Therefore, both divisions and departments can be reconstructed independently from each other. Thus, the matrix hierarchy with minimal cost contains only optimal divisions and departments. So, **the matrix *r*∗-hierarchy** H_{matrix} **has minimal cost**. The cost of this hierarchy is given by:

$$c(H_{matrix}) = \frac{(r_* + 1)^\alpha}{r_* - 1}[l(n-1)(\lambda^\alpha + c_0^\alpha) + n(l-1)(\theta^\alpha + c_0^\alpha)]. \qquad (29)$$

For example, if $\alpha = 2$ then the matrix 3-hierarchy (see Figure 20) has minimal cost. In expression (29) the first item corresponds with total cost of *l* divisions, the second one corresponds with total cost of *n* departments. The internal flow of the top manager of the matrix hierarchy equals to zero. Therefore, expression (26) implies that his or her cost equals to zero too.

The following key proposition of this chapter examines optimality of divisional, functional and matrix hierarchies.

Proposition 5. *For process lines with functional links and cost function* (26) *there exists the optimal divisional, functional or matrix hierarchy.*

The proposition implies that one of typical hierarchies (divisional, functional or matrix) has minimal cost among all hierarchies from set $\Omega(N)$. Thus, **it is not necessary to consider more complex hierarchies; it is enough to compare typical hierarchies, which are often used in real firms**.

For example, in Figure 25 non-optimal hierarchy is shown. This hierarchy is more complex than typical hierarchies. There are two divisions in the hierarchy. Division heads are immediately subordinated to strategic manager m_2 controlling functional interactions between divisions. Also there are three departments in the hierarchy. Heads of the departments are immediately subordinated to strategic manager m_1 controlling product interactions between departments.

In Figure 25 three product flows inside the third process line are controlled neither by the strategic manager nor by divisions. These flows are controlled by individual managers m_3 and m_4 that are not part of a division. Similarly three functional flows are controlled by individual managers m_5, m_6 and m_7. Each of them is not part of a department. The top manager immediately controls managers m_1-m_7. To simplify Figure 25 the top manager is not shown.

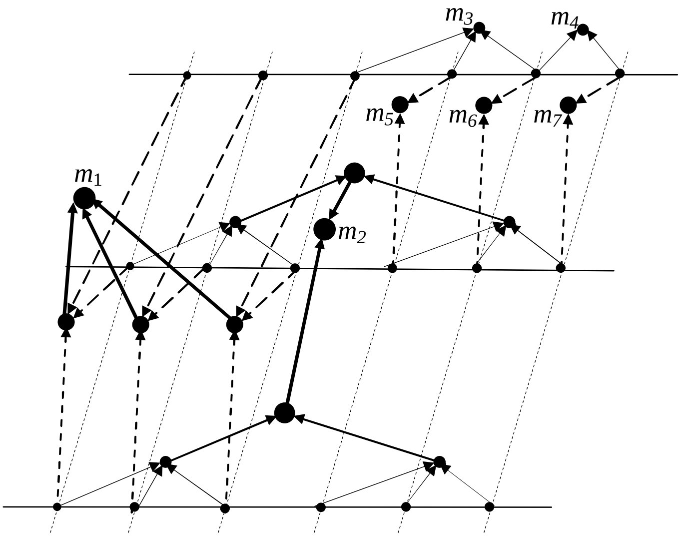

Figure 25. An Example of Non-Typical Hierarchy
(the top manager is not drawn)

Proposition 5 implies that the cost of such hierarchy as shown in Figure 25 is greater than or equal to the minimal cost of typical hierarchies. The proof of Proposition 5 bases on comparison of the cost of any optimal hierarchy with costs of typical hierarchies.

In the next section we compare costs of typical hierarchies. It allows to obtain the optimal hierarchy for different values of parameters of the model.

2.8. Divisional, Functional and Matrix Hierarchies Optimality Conditions

Proposition 5 implies that divisional, functional or matrix hierarchy is optimal. In any case **the optimal span of control equals** $\boldsymbol{r_*}$. r_* depends only on environment instability (see Figure 15 on page 44). Extremely unstable environment ($\alpha > 2.5$) minimizes the optimal span of control ($r_* = 2$). If the environment stabilizes then r_* increases. In Figures 18, 19 and 20 there are examples of hierarchies with r_*=3 (for example, for $\alpha = 2$ one of them is optimal). Below we consider only divisional, functional and matrix hierarchies with optimal span of control r_*.

To solve optimal hierarchy problem it is enough to compare minimal costs of typical hierarchies. Using expressions (27)-(29) we can simplify conditions $c(H_{matrix}) \le c(H_{divisional})$, $c(H_{matrix}) \le c(H_{functional})$ and $c(H_{divisional}) \le c(H_{functional})$. So, we obtain the following inequalities:

$$c_0^\alpha \le \theta^\alpha \frac{n^\alpha - n}{n-1},\ c_0^\alpha \le \lambda^\alpha \frac{l^\alpha - l}{l-1} \text{ and } \theta^\alpha \frac{n^\alpha - n}{n-1} \le \lambda^\alpha \frac{l^\alpha - l}{l-1}.$$

Thus, we can obtain optimal hierarchy by comparing values $\theta^\alpha (n^\alpha - n)/(n-1)$, $\lambda^\alpha (l^\alpha - l)/(l-1)$ and c_0^α. If $\theta^\alpha (n^\alpha - n)/(n-1)$ is minimal value then the divisional hierarchy is optimal. If $\lambda^\alpha (l^\alpha - l)/(l-1)$ is minimal value then the functional hierarchy is optimal. If c_0^α is minimal value then the matrix hierarchy is optimal.

This result can be shown graphically (see Figure 26). The matrix hierarchy optimality conditions are given by:

$$\frac{c_0}{\theta} \le \left(\frac{n^\alpha - n}{n-1}\right)^{\frac{1}{\alpha}} \text{ and } \frac{c_0}{\lambda} \le \left(\frac{l^\alpha - l}{l-1}\right)^{\frac{1}{\alpha}}. \tag{30}$$

The condition $c(H_{divisional}) \le c(H_{functional})$ is given by:

$$\frac{c_0}{\lambda}\left(\frac{n^\alpha - n}{n-1}\right)^{\frac{1}{\alpha}} \le \frac{c_0}{\theta}\left(\frac{l^\alpha - l}{l-1}\right)^{\frac{1}{\alpha}}. \tag{31}$$

Using expressions (30) and (31) we can draw the optimality diagram with abscissa axis c_0/θ and axis of ordinates c_0/λ. Thus, **the abscissa corresponds to the ratio between fixed and variable cost of one functional link control in the stable environment**. **Similarly the ordinate corresponds to the ratio between fixed and variable cost of one product link control**. So, optimality of divisional, functional or matrix hierarchy does not depend on “scale” (unit of measurement) of cost, but depends on ratios between fixed and variable cost.

Expressions (30) imply that the matrix hierarchy optimality region is located at the left below the point with coordinates $([(n^\alpha - n)/(n-1)]^{1/\alpha};[(l^\alpha - l)/(l-1)]^{1/\alpha})$. Consider the line with this point and the coordinate origin (0;0). Expression (31) implies that below this line the cost of the divisional hierarchy is less than the cost of the functional hierarchy. So, divisional, functional or matrix hierarchy optimality diagram looks like Figure 26.

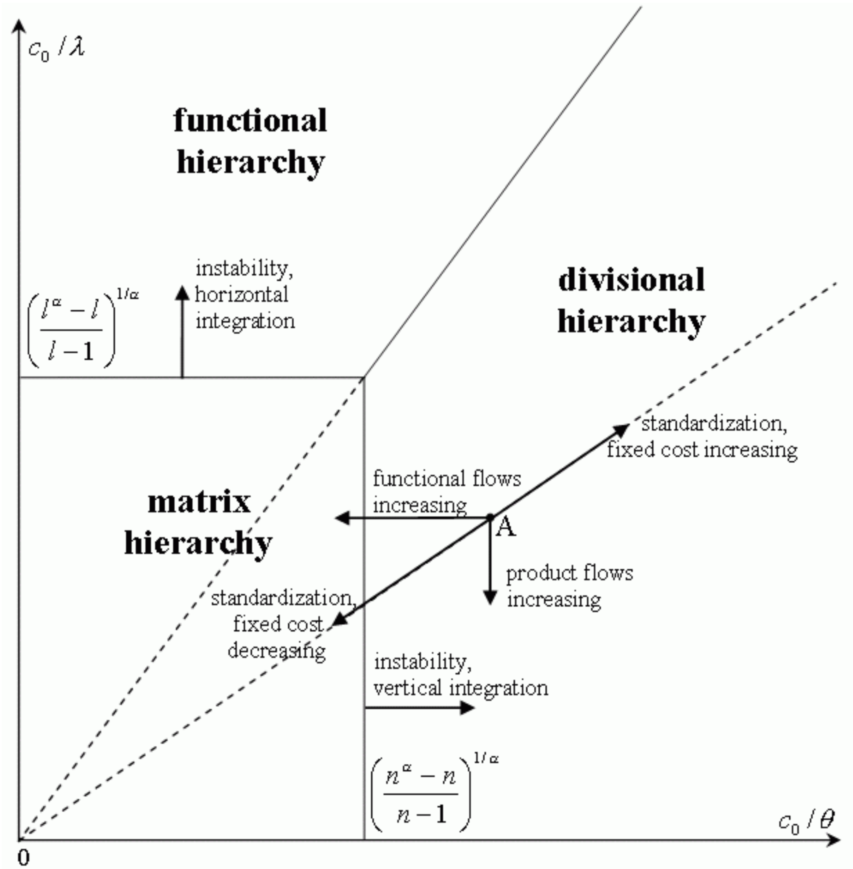

Figure 26. Divisional, Functional and Matrix Hierarchies Optimality Regions

Let's consider the case with the same numbers of process and functional lines $n = l$. In this case in Figure 26 the angle of the boundary line (between optimality of divisional and functional hierarchies) equals 45°. Costs of the divisional and the functional hierarchies depend on product and functional flows intensity λ and θ. If product flows have more intensity than functional flows ($\lambda > \theta$) then the divisional hierarchy is preferable than the functional hierarchy and vice versa. Therefore, we prove the following general rule: **middle-tier managers have to control the most intensive flows in order to decrease costs of strategic managers**. This rule is well-known in practice (for example, Mintzberg (1979) argues that managers on low tiers have to control the most complex (most intensive) links because these managers "hide" the complexity from managers on higher tiers). For the case $n = l = 2$ Harris and Raviv (2002) also prove this rule. Figure 26 shows that the matrix hierarchy is optimal if both product and functional flows intensity is too high. So, in this case middle-tier managers have to control all technological flows and to "hide" the complexity from the top manager.

The following lemma defines the behavior of boundary lines of matrix hierarchy optimality region.

Lemma 6. If $n \geq 2$ and $\alpha > 1$ then value $[(n^{\alpha} - n)/(n-1)]^{1/\alpha}$ monotonously increases both with n and with α.

Also the lemma implies that value $[(l^{\alpha} - l)/(l-1)]^{1/\alpha}$ monotonously increases too. Therefore, if n or α increases then the vertical boundary shifts to the right. Similarly if l or α increases then the horizontal boundary shifts upward.

Let's consider the case with the same intensity of product and functional flows: $\lambda = \theta$. In Figure 26 this point belongs to the line with the angle 45º (this line divides all diagram into halves). If $n > l$ then the cost of the functional hierarchy is less than the cost of the divisional hierarchy and vice versa[48]. So, **if intensity of product and functional flows is the same then middle-tier managers have to control shorter lines to decrease intensity of the flows controlled by strategic managers**. Indeed, if $n > l$ then the functional hierarchy (see Figure 19) is preferable because middle-tier managers control shorter lines and strategic managers control interactions with intensity $l\lambda$ between functional lines. In this case in the divisional hierarchy (see Figure 18) strategic managers control interactions with intensity $n\theta = n\lambda > l\lambda$

[48] If $n>l$ then the angle of the boundary line between regions of optimality of divisional and functional hierarchies is less than 45°, if $n<l$ then the angle is greater than 45°.

between "long" process lines. Similarly if $n < l$ then the divisional hierarchy is preferable than the functional hierarchy.

Suppose the firm grows "in both directions" (both l and n increases). In this case the matrix hierarchy optimality region expands (see Lemma 6 and Figure 26). **If *n* and *l* increase than costs of strategic managers both in the divisional and in the functional hierarchy increase too. Therefore, the matrix hierarchy becomes optimal**.

Let's note that large increase of n and l can be compensated by small decrease of flows' intensity. For large n and l boundary lines of matrix hierarchy optimality region increase as $n^{(\alpha-1)/\alpha}$ and $l^{(\alpha-1)/\alpha}$. If the environment is too unstable ($\alpha = 2$) then double increase of n and l (four times increase of the number of workers) is compensated by $\sqrt{2} \approx 1.4$ times decrease of flows' intensities θ and λ. In this case in Figure 26 the point $(c_0 / \theta; c_0 / \lambda)$ shifts to the right upward proportionally with boundary lines of matrix hierarchy optimality region. It does not change the type of the optimal hierarchy. If the environment stabilizes then small decrease of flows' intensity compensates even higher increase of the size of the firm. For example, if $\alpha = 1.1$ then double increase of n and l is compensated by 7% decrease of intensities.

The described rule may be interpreted as limits of growth of the firm with tree-like hierarchy. If the environment is quite stable then the firm with divisional or functional hierarchy can grow infinitely. In real cases with instability **the growth of the firm with tree-like hierarchy is limited** because increase of strategic managers' costs causes hire of additional middle-tier managers controlling all flows (the matrix hierarchy becomes optimal).

Lemma 6 implies that **if environment instability α increases then the matrix hierarchy becomes optimal** (see Figure 26). Management science (see, for instance, Mintzberg (1979)) argues that unstable environment leads to the matrix hierarchy. This empiric dependence can be explained using the introduced model. In unstable environment strategic managers can not control many flows and hire middle-tier managers to control it. If environment instability is too large then boundary lines of matrix hierarchy optimality region tend to n and l. So, in extremely unstable environment the matrix hierarchy is optimal for any reasonable ratios between fixed and variable cost[49].

On the contrary in the stable environment the matrix hierarchy is not optimal. If $\alpha = 1$ then boundaries of matrix hierarchy optimality region equal to zero (see Figure 26). Moreover, in stable environment the cost of the divisional and the functional hierarchy is the same. In this case optimal span of control infinitely increases (see Figure 15 on page 44). Thus, either divisions or departments are two-tier hierarchies and the single strategic manager controls their interactions.

Let's consider dependence between standardization and the type of optimal hierarchy. In Section 2.2 we argue that standardization increase decreases intensity of both product and functional flows that must be controlled by managers. Therefore, standardization increase proportionally decreases λ and θ.

Consider point *A* in divisional hierarchy optimality region in Figure 26. Standardization increase shifts point *A* along the line far from the coordinate origin. In Figure 26 this shift is denoted by the arrow to the right upward of *A*. Therefore, standardization increase does not change optimality of the divisional hierarchy. On the contrary, standardization decrease shifts the point in matrix hierachy optimality region. Similarly we can consider a point in functional hierarchy optimality region. So, we obtain the following results. Standardization increase does not change optimality of the divisional or functional hierarchy. Standardization decrease leads to optimality of the matrix hierarchy.

The following empirical dependence is well-known: in real firms the matrix hierarchy is preferable for little standardization (see, for instance, Mintzberg (1979)). The introduced model explains this empiric dependence in the following way. Little standardization causes large costs of strategic managers. To reduce these costs it is necessary to increase number of middle-tier managers that control flows.

If the parameters of the model change then the type of optimal hierarchy can change too (see regions on page 26). In this case the hierarchy in the firm becomes non-optimal and we have to *restructure* the

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49 Fixed cost of one link control is less than or equal to variable cost of control all links inside a process line or a functional line.

firm (have to change non-optimal hierarchy)[50]. Usually the restructure requires large money and time. Therefore, it is interesting to analyze stability of the optimal hierarchy with respect to change of key parameters of the model.

The matrix hierarchy is stable with respect to decrease of standardization and environment stability. Increase of these parameters causes restructure of the matrix hierarchy.

The divisional hierarchy and the functional hierarchy are stable with respect to increase of standardization and environment stability. Decrease of these parameters causes optimality of the matrix hierarchy.

Similarly we can consider the type of optimal hierarchy for different fixed cost c_0. Figure 26 shows that c_0 change causes the same effects as standardization change. Therefore, the matrix hierarchy is optimal for small fixed cost. Either the divisional hierarchy or the functional hierarchy is optimal for large fixed cost. Converse statements hold for proportional change of variable costs θ and λ. So, the divisional hierarchy and the functional hierarchy are stable with respect to increase of the ratio between fixed and variable cost. Decrease of this ratio causes optimality of the matrix hierarchy. Harris and Raviv (2002) prove similar dependence between decrease of the fixed cost and optimality of the matrix hierarchy. Let's explain this dependence. A strategic manager controls total flow between divisions or departments without taking into account details of flows between individual workers. Thus, from his or her point of view there is one link with large intensity between two "nearest-neighbor" divisions or departments. Otherwise, middle-tier managers must control many individual links with small intensity. Therefore, if fixed cost increases then costs of middle-tier managers increase more than costs of strategic managers. In this case the matrix hierarchy with maximal number of middle-tier managers becomes non-optimal.

[50] Without restructuring the firm can lose competition because its effectiveness is less than the effectiveness of competitors with optimal hierarchies.

Let's define *two types of growth of the firm*:

1. *Horizontal integration* corresponds with increase of the number l of process lines. The firm can buy similar plants located in other regions, producing similar products, etc. For example, an oil-processing company can buy one more refinery to increase volume of output or to occupy new regional market.

2. *Vertical integration* corresponds with increase of the length n of process lines. The firm can buy vendors or customers. As a result, the length of the whole process line increases (the number of operations from purchasing raw materials to finished product increases). For example, an oil-processing company can buy oil-production firms and gasoline stations to control all process line from oil production to ultimate customer.

Many examples of horizontal and vertical integration are described in managements science literature. Let's explore dependences between different types of integration and the type of the optimal hierarchy. Suppose there is optimal divisional hierarchy in the firm (there is some point in the region of divisional hierarchy optimality). Horizontal integration increases l and expand the regions of divisional and matrix hierarchy optimality (see Figure 26). Therefore, after horizontal integration the divisional hierarchy remains optimal. Vertical integration narrows the region of divisional hierarchy optimality. It can cause restructure of the initial divisional hierarchy to functional or matrix hierarchy. Thus, for the firm with divisional hierarchy horizontal integration is more reasonable because vertical integration can cause restructure.

Let's consider dependences between the type of the optimal hierarchy and change of flows. Increase of product flows corresponds with production volume increase. If we can increase production volume with no increase of functional flows then in Figure 26 point A shifts downward. In this case the divisional hierarchy remains optimal because only costs of middle-tier managers increase, but costs of strategic managers do not change. On the contrary, increase of functional flows increases costs of strategic managers. It can cause restructure (point A in Figure 26 shifts to the left). So, we prove the following facts.

The divisional hierarchy is stable with respect to horizontal integration and increase of production volume with no functional

flows increase. Vertical integration and increase of functional flows cause restructure of the divisional hierarchy.

Similarly we can consider the firm with optimal functional hierarchy (in Figure 26 there is some point in the region of functional hierarchy optimality). So, we prove the following facts.

The functional hierarchy is stable with respect to vertical integration and increase of functional flows. Horizontal integration and increase of production volume (product flows) cause restructure of the functional hierarchy.

On the whole Chapter 2 describes the cost function and proves that one of typical hierarchies is optimal for this function. Typical hierarchies are often used in practice. Therefore, the main proposition of this chapter – optimality of divisional, functional or matrix hierarchy – corresponds with many real firms. It allows to model many dependences between the type of optimal hierarchy and environment instability, standardization, intensities of product and functional flows, horizontal and vertical integration, etc. All these dependences are modeled using the example (26) of sectional cost function[51]. So, Chapter 2 shows that exploration of the class of sectional functions may be useful for modeling real firms. Such exploration is described below in Chapter 3.

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[51] Manager's cost depends only on sets of workers controlled by immediate subordinates. See the formal definition on page 79.

3. General Model

In this chapter we consider the hierarchy optimization problem for arbitrary sectional cost function. So, the manager's cost is given by $c(s_H(v_1),\ldots,s_H(v_k))$, where $s_H(v_1),\ldots,s_H(v_k)$ are groups controlled by immediate subordinates of the manager[52].

In Chapters 1 and 2 special sectional functions (depending on flows) were considered. There was shown that using such cost functions we can model many practical effects occurring in real firms and described in management science. Below we give some other interesting examples of sectional functions that do not depend on flows (for example, manager's cost depends on type of interaction between immediate subordinates). So, using sectional functions we can model various optimal organizational hierarchy problems.

The class of all sectional functions is also interesting from the mathematical point of view: any anonymous (with respect to manager's permutation) and additive (with respect to addition of managers) hierarchy cost function is sectional (see Section 3.1).

For arbitrary sectional function optimal hierarchy problem is very complicated. But in some cases we can find optimal hierarchy for wide classes of sectional functions using methods of this chapter. The methods can be used to research various applications.

Section 3.1 defines the sectional function and considers some interpretations. Sections 3.2, 3.3 and 3.4 contain theoretical methods for solving optimal hierarchy problem for special classes of sectional functions. Examples of sectional function are described in Section 3.5. We find optimal hierarchy for these examples using theoretical methods. In Section 3.6 we introduce method to find the tree with minimal cost. Section 3.7 analyses optimal hierarchy that control several given groups of workers. In this case optimal hierarchy problem is much more complex, but for some sectional functions we can solve it using the methods of Sections 3.3 and 3.4.

[52] See the formal definition of sectional function on page 79.

3.1. Definition of Sectional Cost Function

Definition 7. *Cost function of the manager* $m \in M$ *in the hierarchy* $H = (N \cup M, E) \in \Omega(N)$ *is called sectional if it is given by*:

$$c(s_H(v_1),\ldots,s_H(v_k)), \tag{32}$$

where $v_1,\ldots,v_k$ *are all immediate subordinates of the manager* m, $s_H(v_1),\ldots,s_H(v_k)$ *are groups controlled by employees* $v_1,\ldots,v_k$, $c(\cdot)$ *is a non-negative real function of set of groups.*[53] *Cost of total hierarchy equals to total managers' costs*[54]:

$$c(H) = \sum\nolimits_{m \in M} c(s_H(v_1),\ldots,s_H(v_k)) . \tag{33}$$

Hierarchy's cost function (33) *is also called sectional.*

Definition 7 generalizes Definitions 5 and 6 (see Section 1.6) because in Definition 7 we do not specify the function $c(s_H(v_1),\ldots,s_H(v_k))$. So, the basic model is generalized.

Manager's cost (32) depends only on the groups of workers ("section") controlled by his or her immediate subordinates.

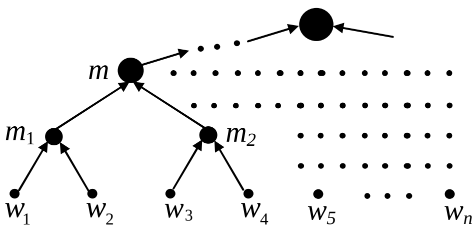

Figure 27. Part of Hierarchy

Let's explain Definition 7 using an example (see the part of hierarchy in Figure 27). The manager m controls the group $\{w_1,w_2,w_3,w_4\}$ with the help of two subordinated managers m_1 and m_2. Managers m_1

[53] The function $c(\cdot)$ depends on the set $\{s_H(v_1),\ldots,s_H(v_k)\}$ of groups and does not depend on order of these groups. So, the manager's cost does not depend on numeration of his or her immediate subordinates $v_1,\ldots,v_k$. Some of groups $s_H(v_1),\ldots,s_H(v_k)$ may be the same. In this case the "set" $\{s_H(v_1),\ldots,s_H(v_k)\}$ contains repeated elements.

[54] In expression (33) and below the symbol $c(\cdot)$ denote both manager's and hierarchy's cost.

and m_2 control the groups $\{w_1,w_2\}$ and $\{w_3,w_4\}$ respectively. Suppose managers m_1 and m_2 cope with controlling of the subordinated employees. In this case the cost of the manager m does not depend on controlling methods inside the groups $\{w_1,w_2\}$ and $\{w_3,w_4\}$. For example, the managers m_1 and m_2 can control subordinated workers immediately or with the help of some subordinated managers. It is of no importance for manager's m cost because direct interactions between m and workers are not necessary.

Definition 7 implies that **the cost of the manager depends only on division of subordinated group of workers between immediately subordinated employees**. In the example noted above the group $\{w_1,w_2,w_3,w_4\}$ is divided into non-overlapping subgroups $\{w_1,w_2,w_3,w_4\}=\{w_1,w_2\} \cup \{w_3,w_4\}$. So, the cost of the manager m equals $c(\{w_1,w_2\},\{w_3,w_4\})$. Thus, we suppose that the cost of a manager depends only on the "*section*"[55] controlled by the manager immediately. In Figure 27 the "section" of the manager m consists of m and subordinated managers m_1 and m_2. The cost of the manager does not depend on other parts of the hierarchy.

Moreover, we suppose that the cost of the manager depends only on the "quantity" of administrative labor (for example, planning and monitoring). So, the cost of the manager does not depend on individual efficiency of managers. Thus, a sectional function does not change with any permutation of the managers with no modification of subordination edges. So, a sectional function is *anonymous* with respect to manager's permutation. Also manager's cost does not depend on numeration of his or her immediate subordinates. For example, in Figure 27 the manager's m cost depends only on the set $\{\{w_1,w_2\},\{w_3,w_4\}\}$ of groups (the equality $c(\{w_1,w_2\},\{w_3,w_4\})=c(\{w_3,w_4\},\{w_1,w_2\})$ holds).

Definition 7 implies that a sectional function is *additive*: a hierarchy cost equals to total costs of all managers in the hierarchy.

Mishin and Voronin (2003), Mishin (2003b) consider arbitrary cost function depending on hierarchy[56]. Anonymity and additivity

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[55] For example, department, division or some over business unit.

[56] In other words an arbitrary function $c:\Omega \to R_+$ is considered (the applicable domain of the non-negative real function $c(\cdot)$ is the set of all hierarchies).

conditions are generalized for such functions. It is proven that **any anonymous and additive hierarchy cost function is sectional**.

Hence the class of all sectional functions is too wide. Using of sectional functions we can model various optimal hierarchy problems (see Chapters 1 and 2 and examples below). So, we can suppose that sectional functions are useful to model organizational hierarchies in firms. Therefore, it is important to research sectional functions and find the optimal hierarchies. Some results are described below in this chapter. Generally the cost of the manager can depend on individual efficiency, hierarchical tier, superiors or the whole hierarchy. Such cost functions are not sectional and not considered in this paper.

In Definition 7 some of the groups $s_H(v_1),\ldots,s_H(v_k)$ can be the same or nested one into another. Suppose $s_H(v_1) \subseteq s_H(v_2)$. So, the employee v_1 controls part of the group subordinated to the employee v_2. Thus, one immediate subordinate of the manager m duplicates part of the labor of another immediate subordinate. In the basic model such duplication does not reduce manager's m cost (see Lemma 4 on page 23).

Below we consider only sectional functions satisfying the condition of Lemma 4. So, if $s_H(v_1) \subseteq s_H(v_2)$ then the following inequality holds:

$$c(s_H(v_2),\ldots,s_H(v_k)) \leq c(s_H(v_1),\ldots,s_H(v_k)).$$

For example, "auxiliary" immediate subordinate v_1 can waste manager's m time discussing some problems inside the group $s_H(v_2)$ (such problems are completely controlled by the manager v_2). So, we can remove subordination edge (v_1,m) with no increase of manager's m cost. After removal costs of other managers do not change because groups controlled by all managers do not change.

The proof of Proposition 1 holds true for any sectional function satisfying condition of Lemma 4. Therefore, Proposition 1 (page 24) holds true for concerned sectional functions. So, **we can consider only hierarchies satisfying conditions (i)-(iii) of Proposition 1**: all managers control different groups of workers, all employees are subordinated to the single top manager, any manager's immediate subordinates do not control one another. All optimal hierarchies obtained below satisfy conditions (i)-(iii).

Below in some cases the sectional cost function is given by simplified notation $c(s_1,\ldots,s_k)$ instead of $c(s_H(v_1),\ldots,s_H(v_k))$. *The value of the function $c(s_1,\ldots,s_k)$ corresponds with the cost of some manager with immediate subordinates controlling the groups $s_1,\ldots,s_k$.*

3.2. Tree Optimality Condition

As it is shown in Section 1.9 (see example on page 27), in some cases there is no optimal hierarchy among the trees. However, in other cases the minimal cost tree can be optimal. For example, the tree is the optimal hierarchy controlling the symmetric process line (see Section 1.10). The tree (divisional or functional) can be optimal hierarchy controlling the process lines with functional links (see Section 2.8). Moreover, the trees are typical organizational hierarchies in many firms. Therefore, it is important to obtain the conditions when some tree is the optimal hierarchy. Below we consider the sufficient condition for tree optimality. This is the so-called group-monotonity condition.

Definition 8. *A sectional cost function is called group-monotonic if the manager's cost does not decrease with the expansion of the groups controlled by the immediate subordinates and with the addition of new immediate subordinates. So, for any groups $s_1,\ldots,s_k$ the following inequalities hold:*

$c(s_1,s_2,\ldots,s_k) \le c(s,s_2,\ldots,s_k)$, *where s contains s_1 ($s_1 \subset s$);*

$c(s_1,s_2,\ldots,s_k) \le c(s,s_1,\ldots,s_k)$, *where s is any group.*

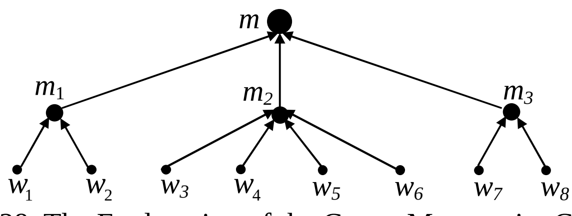

Figure 28. The Explanation of the Group-Monotonity Condition

Let's explain Definition 8 by the example. Let the top manager m be the chief of the hierarchy shown in Figure 28. Let the immediately subordinated managers m_1, m_2 and m_3 control the supply department (the group $\{w_1,w_2\}$), the manufacturing department (the group $\{w_3,w_4,w_5,w_6\}$) and the marketing department (the group $\{w_7,w_8\}$) respectively.

The chief m controls all the firm with the help of his or her immediate subordinates m_1, m_2 and m_3. Thus, m communicates with the heads of the departments m_1, m_2 and m_3 to solve interdepartment interaction problems. Also the manager m can solve part of problems inside each of the departments. Therefore, the cost of the manager m can consist of the following two parts.

1. The first part of the manager's m cost can be concerned with the controlling of immediate subordinates' interactions. Let this part depend on the number of immediate subordinates. Thus, the first part of the cost is determined by some non-decreasing function $\chi(\cdot)$. The cost $\chi(3)$ of the manager m corresponds to the controlling of managers' m_1, m_2 and m_3 interactions. The function $\chi(\cdot)$ type depends on the business area, possible communication mechanisms between the manager and his or her immediate subordinates, etc. Consider an example. Suppose the heads of the departments usually communicate with each other directly. In the case of communication problems the heads of the departments resort to the help of the immediate superior. Let the communication problem appear with the probability $0<\delta_1<1$. If only paired communications are possible then the first part of the manager's cost can be given by $\chi(k)=x_1\delta_1 k(k-1)/2$, where x_1 is the average cost of solving one communication problem, k is the number of immediately subordinated employees, $k(k-1)/2$ is the number of their paired communications. So, the first part of the chief's m cost may be equal $3x_1\delta_1$ (see Figure 28). If communication problems can appear between three, four and more employees then the function $\chi(\cdot)$ can grow exponentially.

2. The second part of the manager's m cost can be concerned with problems inside the groups controlled by the immediate subordinates. For example, the manager can perform some administrative labor when any subordinated worker is dismissed (interview with new worker, signature of some documents, etc.). Thus, the second part of the cost depends on the number of subordinated workers. So, the second part of the manager's cost is determined by some non-decreasing function

$\varsigma(|s_1 \cup \ldots \cup s_k|)$ depending on the size of the group $s_1 \cup \ldots \cup s_k$, where $s_1,\ldots,s_k$ are the groups controlled by immediate subordinates of the manager. The type of function $\varsigma(\cdot)$ depends on the business area, manager's responsibility, etc. Consider an example. Let an employee be dismissed with the probability $0<\delta_2<1$. Let x_2 be the average manager's cost in case of one dismissal. Then the second part of the manager's cost can be given by $x_2\delta_2|s_1 \cup \ldots \cup s_k|$. So, the second part of the chief's m cost may be equal $8x_2\delta_2$ (see Figure 28).

Thus, sectional function can be given by:

$$c(s_1,\ldots,s_k) = \chi(k) + \varsigma(|s_1 \cup \ldots \cup s_k|) \quad (34).$$

Let's include the worker w_3 into the supply department. Now w_3 is immediately subordinated to the managers m_1 and m_2 (w_3 is a member of both supply and manufacturing departments). So, the supply department is expanded. The manager m_1 controls the expanded group $\{w_1,w_2,w_3\}$ instead of the initial group $\{w_1,w_2\}$. Thus, the chief's m cost equals $c(\{w_1,w_2,w_3\},\{w_3,w_4,w_5,w_6\},\{w_7,w_8\}) = \chi(3)+\varsigma(8)$. The cost does not change. Similarly function (34) does not decrease with any expansion of the groups $s_1,\ldots,s_k$. So, cost function (34) satisfies the first condition of Definition 8.

Let's change the hierarchy in Figure 28 in some other way. We hire three new workers w_9,w_{10},w_{11} and manager m_4. Then we organize the fourth department consisting of these employees (new department consists of new workers immediately subordinated to the manager m_4). Finally we immediately subordinate the manager m_4 to the chief m. So, the chief's m cost is given by:

$$c(\{w_1,w_2\},\{w_3,w_4,w_5,w_6\},\{w_7,w_8\},\{w_9,w_{10},w_{11}\}) = \chi(4)+\varsigma(11).$$

Thus, the chief's m cost does not decrease. Similarly function (34) (manager's cost) does not decrease with any addition of new immediate subordinates[57]. So, cost function (34) satisfies the second condition of Definition 8.

[57] In some practical situations a manager can decrease his or her cost by increasing the number of immediately subordinated managers ("assistants"). However, if the "assistants" coordination cost is sufficiently high then it is reasonable to model the firm with the help of group-monotonic function.

Therefore, function (34) is group-monotonic. This group-monotonic function and some other examples below can correspond with the cost of the manager in real firm. It is interesting to obtain an optimal hierarchy in the case of group-monotony. The main result is given by the following proposition.

Proposition 6. *If sectional cost function is group-monotonic then there exists optimal tree*.

According to this proposition **if cost function is group-monotonic then optimal hierarchy can be found among the trees**. So, in the optimal hierarchy the immediate subordinates of common manager control non-overlapping groups of workers. The immediate subordinates of common manager do not "duplicate" each other administrative labor.

Therefore, to find optimal hierarchy we can verify the inequalities of Definition 8. If these inequalities hold then we can consider only the trees. In this case optimal hierarchy problem is much simpler.

Using Proposition 6 we can find the type of the whole optimal hierarchy by means of manager's cost function analysis (inequalities verification).

The cost function in the basic model (see Definition 5 on page 21) is not group-monotonic as illustrated by the following example. Let the workers $w_1,\ldots,w_8$ in Figure 28 be linked with some process line. Manager's m internal flow equals to the sum of $f(w_2,w_3)$ and $f(w_6,w_7)$. Let's subordinate the worker w_3 to the manager m_1. The group controlled by the manager m_1 is extended. So, instead of m the manager m_1 controls the flow $f(w_2,w_3)$. Therefore, manager's cost can be reduced after expansion of the group controlled by the immediate subordinate. It contradicts Definition 8.

Thus, basic model cost function is not group-monotonic. But in some cases the tree with minimal cost is optimal (for example, in the case of symmetric process line, Section 1.10). So, **the group-monotonity is sufficient condition but not requirement for the tree optimality**.

Proposition 6 implies that if the cost function if group-monotonic then we only need to find minimal cost tree. This tree is the optimal hierarchy.

Minimal cost tree can be found using the algorithms developed by Mishin and Voronin (2001, 2003). For an arbitrary sectional cost function the exact algorithm's complexity is too high (the minimal cost tree can be found only for 15-20 workers[58]). Consider the cost function given by the expression $c(|s_1|,\ldots,|s_k|)$. So, the manager cost depends only on the span of control k (number of immediate subordinates) and on the numbers $|s_1|,\ldots,|s_k|$ of workers in the groups controlled by the immediate subordinates (but not on individual workers in these groups!). In this case the exact algorithm finds the minimal cost tree for 70-100 workers. For example, function (34) can be given by $c(|s_1|,\ldots,|s_k|)$. So, for cost function (34) the algorithm finds an optimal hierarchy if the number of workers is less than or equal to 100.

Mishin and Voronin (2001) also prove that it is impossible to sufficiently reduce exact algorithm's complexity. Therefore, in the paper noted above some heuristic algorithms are developed. These algorithms have much less complexity and find trees with approximately minimal cost. For arbitrary function given by $c(|s_1|,\ldots,|s_k|)$ two heuristic algorithms are developed. Their complexities grow as n^2 and n^3.

If the cost function is group-monotonic then optimal hierarchy problem can be solved using the algorithms. For other cost functions the tree obtained by the algorithms may be non-optimal hierarchy. But this tree is useful, for example, to compare the best tree with actual hierarchy in the firm.

Because of some reasons we can consider only r-hierarchies (span of control or number of manager's immediate subordinates is less than or equal to r) [59]. Noted above algorithms can find r-tree with minimal cost. For fixed r the algorithms' complexity is much less. If the cost function is group-monotonic then the tree obtained by the algorithms

[58] Using personal computer in several minutes.

[59] For example, in some firm a manager cannot control more than 10 immediate subordinates. In this case we can consider the cost function with infinite value for 11 or more immediate subordinates. But it may be difficult to investigate this function. So, it is easier to consider only 10-hierarchies.

has minimal cost among all r-hierarchies (it can be proved similarly to the proof of Proposition 6).

In the following section we consider minimal and maximal span of control optimality conditions.

3.3. 2-Hierarchy and Two-Tier Hierarchy Optimality Conditions

In this section we consider narrowing and widening conditions. If cost function is narrowing (widening) then we can decrease (increase) number of any manager's immediate subordinates with no hierarchy cost increase. Many sectional functions are narrowing or widening (see examples in Section 3.5). Therefore, it is important to find optimal hierarchies for such functions.

Definition 9. *Sectional cost function is narrowing if for any manager m with immediately subordinated employees $v_1,\dots,v_k$, $k \ge 3$ it is possible to resubordinate several employees from $v_1,\dots,v_k$ to new manager m_1 and immediately subordinate m_1 to the manager m with no hierarchy cost increase. Sectional cost function is widening if any such resubordination does not decrease the cost of a hierarchy.*

Let's explain Definition 9. In Figure 29a) manager m has three or more immediate subordinates $v_1,\dots,v_k$. Consider a narrowing cost function. With no hierarchy cost increase we can hire new immediate superior m_1 for j ($1 < j < k$) employees from $v_1,\dots,v_k$. After the hire the manager m controls these employees with the help of new manager m_1 but not immediately. For example, the result of employees $v_1,\dots,v_j$ resubordination is shown in Figure 29b).

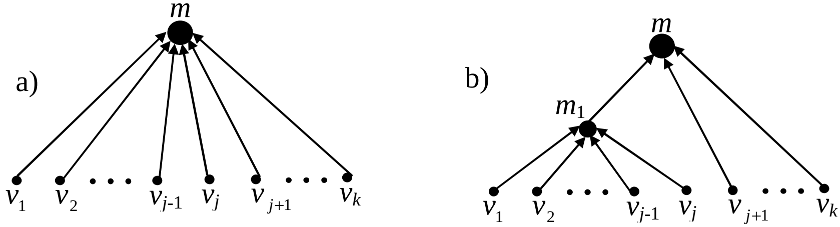

Figure 29. Resubordination for Narrowing or Widening Cost Function

Generally any j employees can be resubordinated. So, there exists such permutation ($i_1,\ldots,i_k$) of numbers ($1,\ldots,k$) that employees $v_{i_1},\ldots,v_{i_j}$ are resubordinated. If cost function is narrowing then for any groups $s_1=s_H(v_1),\ldots,s_k=s_H(v_k)$ controlled by employees $v_1,\ldots,v_k$ some of them can be resubordinated with no hierarchy cost increase.

Thus, *definition of narrowing cost function can be written as follows*. For any groups $s_1,\ldots,s_k$, $k\geq 3$ there exist such number $1<j<k$ and permutation ($i_1,\ldots,i_k$) that the following inequality holds:

$$c(s_1,\ldots,s_k)\geq c(s_{i_1},\ldots,s_{i_j})+c(s_{i_1}\cup\ldots\cup s_{i_j},s_{i_{j+1}},\ldots,s_{i_k})\,. \quad (35)$$

Left-hand member of the inequality is the cost of the manager m before resubordination (see Figure 29a)). Right-hand member of the inequality equals to sum of manager's m_1 cost $c(s_{i_1},\ldots,s_{i_j})$ and manager's m cost $c(s_{i_1}\cup\ldots\cup s_{i_j},s_{i_{j+1}},\ldots,s_{i_k})$ after resubordination (see the example in Figure 29b)). Other managers' costs do not change. So, inequality (35) holds if and only if cost of total hierarchy does not increase.

Inequality (35) may be explained as follows. *To decrease narrowing cost function we can hire manager's m* "*assistant*" m_1 undertaking part of administrative labor. After that the number of manager's m immediate subordinates decreases. So, the hierarchy becomes "narrower" (the span of control decreases).

Consider a widening cost function. Definition 9 leads to the fact that any described above resubordination does not decrease the cost of a hierarchy. So, for any groups $s_1,\ldots,s_k$, $k\geq 3$, any number $1<j<k$ and any permutation ($i_1,\ldots,i_k$) the following inequality holds:

$$c(s_1,\ldots,s_k)\leq c(s_{i_1},\ldots,s_{i_j})+c(s_{i_1}\cup\ldots\cup s_{i_j},s_{i_{j+1}},\ldots,s_{i_k})\,. \quad (36)$$

Inequality (36) may be explained as follows. *For widening cost function it is impossible to decrease the cost of a hierarchy with the help of hiring* "*assistants*".

Consider part of hierarchy in Figure 29b). Let m be the only immediate superior of the manager m_1. In this case we can decrease widening cost function with the help of excess "assistant" m_1 dismissal[60].

[60] After the dismissal the manager m undertakes administrative labor of the manager m_1.

After that the number of manager's *m* immediate subordinates increases. So, the hierarchy becomes "wider" (the span of control increases).

If inequality (35) or (36) is violated on some overlapping groups $s_1,\ldots,s_k$ but held on any non-overlapping groups (i.e. $s_i \cap s_j = \varnothing$ for all $i \neq j$) then cost function is called *narrowing on non-overlapping groups* or *widening on non-overlapping groups* respectively.

Proposition 7. *If sectional cost function is narrowing then there exists optimal 2-hierarchy*.

Corollary (from Propositions 6 and 7). *If sectional cost function is narrowing on non-overlapping groups and group-monotonic then there exists optimal 2-tree*.

Proposition 7 can be proved using described above resubordination for an optimal hierarchy with three or more immediate subordinates of a manager.

The corollary can be proved similarly using resubordination for an optimal tree (Proposition 6 leads to the fact that the optimal tree exists). To resubordinate employees in the tree, only narrowing on non-overlapping groups are required because in a tree groups controlled by immediate subordinates of a manager do not overlap (see Lemma 2 on page 17).

Therefore, to find optimal hierarchy we can verify inequality (35). **If inequality (35) holds then the cost function is narrowing and we can consider only 2-hierarchies with each manager having two immediate subordinates (minimal span of control) because there exists optimal 2-hierarchy**. In this case optimal hierarchy problem is much more easier.

So, using Proposition 7 we can find the type of the whole optimal hierarchy by means of manager cost function analysis (inequality (35) verification). If cost function is group-monotonic then we have to verify inequality (35) only on non-overlapping groups $s_1,\ldots,s_k$. If the inequality holds then the corollary leads to the fact that there exists optimal 2-tree. So, 2-tree with minimal cost is optimal. Minimal cost 2-tree can be found using the algorithms developed by Mishin and Voronin (2001) (see brief description in Section 3.2).

Proposition 8. *If sectional cost function is widening then two-tier hierarchy is optimal.*

Corollary (from Propositions 6 and 8). *If sectional cost function is widening on non-overlapping groups and group-monotonic then two-tier hierarchy is optimal.*

Proposition 8 can be proved using dismissal of excess “assistant” of the top manager until the top manager immediately controls all workers.

The corollary can be proved similarly using dismissal for an optimal tree (Proposition 6 leads to the fact that the optimal tree exists). To dismiss excess “assistant” from the tree, only widening on non-overlapping groups is required because in a tree groups controlled by immediate subordinates of a manager do not overlap.

Therefore, to find optimal hierarchy we can verify inequality (36). **If inequality (36) holds then the cost function is widening and two-tier hierarchy with single manager is optimal (singe manager controls all workers immediately, span of control is maximal)**.

So, using Proposition 8 we can find optimal hierarchy by means of manager cost function analysis (inequality (36) verification). If cost function is group-monotonic then we have to verify inequality (36) only on non-overlapping groups $s_1,\ldots,s_k$. If the inequality holds then the corollary leads to the fact that two-tier hierarchy is optimal. An example of Section 3.5 shows that inequality (36) may hold on non-overlapping groups and violate on overlapping groups. Thus, the corollary can be useful to analyze some sectional cost functions.

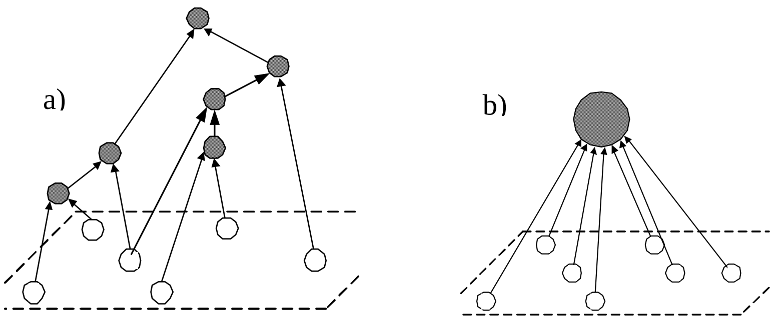

Figure 30. Optimal Hierarchy Examples for Narrowing (a) or Widening (b) Cost Function

Propositions 7 and 8 show that **narrowing functions contrast with widening functions**. Narrowing condition implies 2-hierarchy optimality (see the example in Figure 30a)). 2-hierarchy contains the most number of managers. Each manager performs minimal quantity of administrative labor (controls only two immediate subordinates). The span of control is minimal (equals 2). On the contrary widening condition implies two-tier hierarchy optimality (see the example in Figure 30b)). Two-tier hierarchy contains single manager performing all administrative labor (the manager controls all n worker immediately). The span of control equals n.

Therefore, narrowing and widening conditions imply two extreme cases: minimal and maximal span of control. Most firms have "intermediate"[61] hierarchies (Mintzberg (1979)). But examples in Section 3.5 show that narrowing and widening conditions are very useful because for many sectional cost functions we can obtain narrowing and widening parameter regions (extreme cases). In other parameter regions the function can be used to model most firms.

Consider the cost function $\varphi(\cdot)$ from basic model (see Definition 5 on page 21). So, the manager's cost depends on his or her flows. Suppose function $\varphi(\cdot)$ is subadditive.[62] Let's prove that inequality (36) holds (cost function is widening). Let H_0 be the hierarchy before excess "assistant" m_1 dismissal (see Figure 29b)), H_1 be the hierarchy after m_1 dismissal (see Figure 29a)). Left-hand member of (36) is the cost of the manager m after dismissal. Right-hand member of (36) equals to sum of manager's m_1 cost and manager's m cost before dismissal. Then for the cost function $\varphi(\cdot)$ the following inequalities hold:

$$\varphi(F_{H_0}^{\text{int}}(m_1)+F_{H_0}^{ext}(m_1))+\varphi(F_{H_0}^{\text{int}}(m)+F_{H_0}^{ext}(m))\geq$$
$$\geq\varphi(F_{H_0}^{\text{int}}(m_1)+F_{H_0}^{ext}(m_1)+F_{H_0}^{\text{int}}(m)+F_{H_0}^{ext}(m))\geq\varphi(F_{H_1}^{\text{int}}(m)+F_{H_1}^{ext}(m)).$$

[61] Span of control is more than two and less than n.

[62] For any $x,y\in R_+^p$ the inequality $\varphi(x+y)\leq\varphi(x)+\varphi(y)$ holds.

The first inequality holds because of subadditivity. The second inequality holds because the function $\varphi(\cdot)$ is monotone increasing, internal flow of the manager m in the hierarchy H_1 is less than or equal to sum of managers' m and m_1 internal flows in the hierarchy H_0 ($F_{H_1}^{\text{int}}(m) \le F_{H_0}^{\text{int}}(m) + F_{H_0}^{\text{int}}(m_1)$), external flow of the manager m does not change ($F_{H_1}^{ext}(m) = F_{H_0}^{ext}(m)$). So, inequality (36) holds.

Thus, if $\varphi(\cdot)$ is subadditive then cost function is widening. Proposition 2 (page 26) implies that two-tier hierarchy is optimal for subadditive function $\varphi(\cdot)$. Proposition 8 implies that two-tier hierarchy is optimal for any widening cost function. So, *widening condition generalizes subadditivity condition for all sectional functions*.[63]

Consider interrelation between classes of group-monotonic, narrowing and widening cost functions.

[63] Let's change Definition 5 (page 21) in such way that manager's cost depends only on his or her internal flow. In this case if $\varphi(\cdot)$ is superadditive then cost function is narrowing and if $\varphi(\cdot)$ is subadditive then cost function is widening (Mishin and Voronin (2003)). So, narrowing/widening conditions generalize superadditivity/subadditivity conditions or convexity/concavity conditions (superadditivity/subadditivity are equivalent to convexity/concavity for one-dimensional flows and $\varphi(0) = 0$).

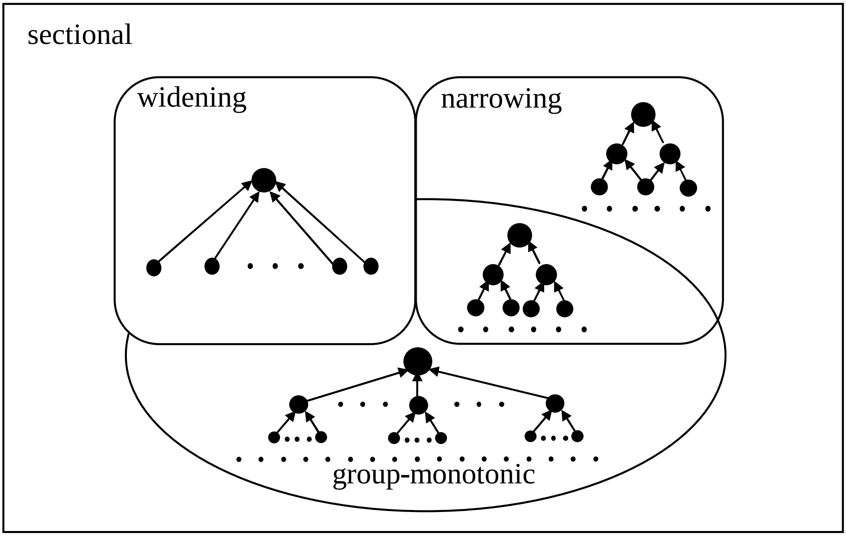

Figure 31. Interrelationship Between Classes of Group-Monotonic, Narrowing and Widening Cost Functions

As noted in Section 3.2 cost function $\varphi(\cdot)$ (depending on flow) is not group-monotonic. Power function $\varphi(\cdot)$ may be widening (because of concavity, see Lemma 5 on page 26), may be neither widening nor narrowing (because of optimal span of control $2 < r_* < +\infty$, see Section 1.11). Also there exist narrowing functions which are not group-monotonic (see Section 3.5).

Examples of Section 3.5 show that group-monotonic cost function may be narrowing, widening or neither narrowing, nor widening. Moreover, in extreme cases a sectional function may be both narrowing and widening.

Interrelationship between classes of group-monotonic, narrowing and widening cost functions is shown in Figure 31. Types of optimal hierarchies for different cases are shown in the figure too (a tree is optimal for group-monotonic functions, two-tier hierarchy is optimal for widening functions, a 2-hierarchy is optimal for narrowing functions, a 2-tree is optimal for group-monotonic and narrowing functions).

In the next section we consider strong narrowing condition and prove optimality of special 2-hierarchy.

3.4. Consecutive Hierarchy Optimality Condition

In Section 3.3 we show that there exists optimal 2-hierarchy for narrowing cost function. In this section we consider particular 2-hierarchies (the so called consecutive hierarchies). Below we define strongly narrowing cost functions. For such functions there exists optimal consecutive hierarchy. Optimization methods described in this section allow to obtain optimal hierarchy for several cost functions (see examples in Section 3.5).

Definition 10. *2-hierarchy is consecutive if any manager in the hierarchy immediately controls at least one worker.*

Similarly with Proposition 1 we can prove the following fact: for any consecutive hierarchy H_1 there exists the consecutive hierarchy H_2 such that $c(H_2) \leq c(H_1)$ and conditions (i)-(iii) of Proposition 1 (see page 24) are satisfied. Thus, there are no managers in H_2 controlling the same group of workers, all managers are subordinated to the single top manager, immediate subordinates of a manager do not control each other. Therefore, among consecutive hierarchies there exists the hierarchy with minimal cost satisfying conditions (i)-(iii). Let's explain the form of hierarchy H_2.

Condition (i) and Definition 10 imply that in H_2 any manager has exactly two immediate subordinates. Top manager m in H_2 controls all workers: $s_{H_2}(m) = N$. Manager m immediately controls some worker w' and some manager m'. Thus, $s_{H_2}(m) = N = s_{H_2}(m') \cup \{w'\}$ (see Lemma 1 on page 16). Condition (iii) implies that manager m' does not control the worker w'. Therefore, $s_H(m') = N \setminus \{w'\}$. Similarly manager m' immediately controls some worker w'' and some manager m'', $s_H(m'') = N \setminus \{w', w''\}$, etc. So, the consecutive hierarchy looks like the hierarchy in Figure 32.

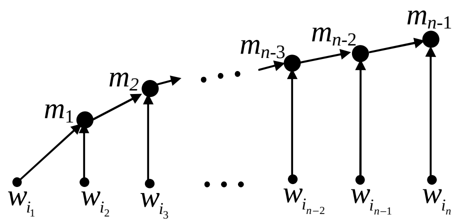

Figure 32. General Form of the Consecutive Hierarchy

The consecutive hierarchy is shown in Figure 32. In any consecutive hierarchy the workers $w_1,\ldots,w_n$ are ordered in some way $w_{i_1},\ldots,w_{i_n}$, where $(i_1,\ldots,i_n)$ is some permutation of numbers $(1,\ldots,n)$. Thus, the consecutive hierarchy has n–1 manager: $M=\{m_1,\ldots,m_{n\text{-}1}\}$ (see Figure 32). The first manager immediately controls the workers w_{i_1} and w_{i_2}. The second manager immediately controls the first manager and the worker w_{i_3}. The third manager immediately controls the second manager and the worker w_{i_4}, etc. The top manager $m_{n\text{-}1}$ immediately controls the worker w_{i_n} and the previous manager $m_{n\text{-}2}$.

Consecutive hierarchies may be interpreted in different ways. Consider several examples.

In a consecutive hierarchy the managers can control quality of conveyorized assembly. Each manager can control quality of some components, semi-finished or finished products. To simplify quality control (to decrease cost) a manager can use results of previous control stages. For example, the manager m_1 can inform the manager m_2 about results of tests of weld seams strength. Using these results the manager m_2 can calculate strength of assembled product. Without these results the manager m_2 must test the whole product strength and quality control cost may increase. Thus, the cost of quality control may depend on order of controlling operations performed by managers. Managers in a consecutive hierarchy controls quality of products after each stage of assembly (after operations of each worker). Therefore, cost of quality control may depend on the permutation $(i_1,\ldots,i_n)$.

Also **a consecutive hierarchy may be interpreted as information processing graph**. Let’s briefly describe several information processing models.

Marschak and Radner (1972) explore the following model of managers processing information incoming from *n* sources. This information may characterize the state of the firm. For example, the sources of this information may be interpreted as workers informing managers about some problems. The managers have to process information and obtain common control action for the whole firm. Information processing may be modeled as calculation of some function of incoming variables (each source is some variable). Associative functions are considered (one of the simplest associative functions is addition). So, the function value does not depend on order of calculations.

In Marschak and Radner model the managers are organized in a tree. Each manager gets information from immediate subordinates, spends some time to calculate value and passes it to immediate superior. Spent time linearly depends on number of immediate subordinates (number of incoming variables). The top manager calculates the final control action. The number of managers and total calculation time (total delay) characterize the tree. It is necessary to obtain the tree with optimal balance between these two characteristics. For example, we can consider some cost function depending on delay and number of managers (Keren and Levhari (1989)). Also we can consider more complex case with periodically repeated information processing. In this case idle managers may begin to process next information before other managers finish to process previous information. So, it is interesting to obtain the minimal cost tree, which copes with processing of all incoming information.

Above cited models are considered in many papers (see, for instance, Keren and Levhari (1983, 1989), Radner (1993), Van Zandt (1996)). Different trees are optimal depending on several conditions. For example, Bolton and Dewatripont (1994) prove that optimal organizational hierarchy may combine a “conveyer belt” type of structure with such tree that only employees on adjacent tiers interact directly.

Thus, it may be interesting to interpret a consecutive hierarchy as information processing graph. In a consecutive hierarchy (see Figure 32) the first manager processes his or her information, then the second manager processes his or her information, etc. So, at any time point only one manager processes information and other managers are idle in this time point.

Therefore, the consecutive hierarchy may correspond with consecutive processing of information incoming from workers. Total calculation time is large for such hierarchy. However, if information incomes periodically then such hierarchy may cope with processing of frequently incoming information.

Associativity of the calculated function leads to the fact that any consecutive hierarchy calculates control action correctly because the value of the function is the same for any permutation $(i_1,i_2,\ldots,i_n)$ (any control order). However, time or cost of processing information from various workers may differ. Thus, **a consecutive hierarchy with minimal cost may correspond with effective consecutive information processing**.

Let's consider *the problem of searching out consecutive hierarchy with minimal cost*. In many cases this problem can be solved analytically (see Section 3.5). A consecutive hierarchy is determined by the permutation $(i_1,\ldots,i_n)$ (see Figure 32). For arbitrary sectional function $n!/2$ consecutive hierarchies may have different costs[64]. However, to obtain optimal hierarchy it is not necessary to compare costs of all these hierarchies. Mishin and Voronin (2002b, 2003) introduce algorithm, which obtain minimal cost consecutive hierarchy with computational complexity 2^n. For arbitrary sectional function this algorithm allows to solve the problem for 30-40 workers[65].

In information processing models it is very interesting to obtain optimal hierarchy, which calculates more then one function. Radner (1992) notes that by now methods solving this problem are unknown. Mishin and Voronin (2002b, 2003) describe algorithm, which obtain minimal cost consecutive hierarchy calculating several functions (see brief description in Section 3.7).

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[64] There exist $n!$ different consecutive hierarchies. But the first and the second workers (see Figure 32) may be permuted with no hierarchy cost change. Thus, $n!/2$ consecutive hierarchies may have different cost. It is easy to construct such sectional function that costs of all these hierarchies differ.

[65] Using personal computer in several minutes.

Consider the sufficient condition for optimality of consecutive hierarchy. It is a strong narrowing condition. If this condition holds then optimal hierarchy problem can be solved analytically or using the cited above algorithm, which gives consecutive hierarchy with minimal cost. Let's define and explain strong narrowing condition.

Definition 11. *Narrowing cost function is strongly narrowing if for any groups s_1, s_2 containing two or more workers at least one of the following conditions hold*:

a) for each $w \in s_1$: $c(s_1, s_2) \geq c(s_1 \setminus \{w\}, s_2) + c((s_1 \setminus \{w\}) \cup s_2, \{w\})$,

b) for each $w \in s_2$: $c(s_1, s_2) \geq c(s_1, s_2 \setminus \{w\}) + c(s_1 \cup (s_2 \setminus \{w\}), \{w\})$.

If the cost function is narrowing then there exists optimal 2-hierarchy H (see Proposition 7). Conditions a) and b) of Definition 11 allow to reconstruct H into optimal consecutive hierarchy. Let's explain this reconstruction using Figure 33.

If in 2-hierarchy H any manager immediately controls at least one worker then this is consecutive hierarchy (see Definition 10). Otherwise consider a manager m with two immediately subordinated managers m_1 and m_2 (see Figure 33a)). If there exist several managers of this type then consider the manager on lowest tier. So, managers m_1 and m_2 immediately control at least one worker. In Figure 33a) manager m_1 immediately controls worker w' and employee v'. Manager m_2 immediately controls worker w'' and employee v''.

Strong narrowing condition (see Definition 11) allows to reconstruct hierarchy shown in Figure 33a) with no cost increase. Let $s_1=s_H(m_1)$ and $s_2=s_H(m_2)$ be the groups controlled by managers m_1 and m_2 respectively. Then employee v' controls group $s_1 \setminus \{w'\}$ and employee v'' controls group $s_2 \setminus \{w''\}$.[66]

[66] Condition (i) of Proposition 1 leads to the fact that all employees control different groups of workers. Therefore, employee v' can not control worker w' because otherwise employee v' and manager m_1 control the same groups. Thus, $s_H(v') = s_1 \setminus \{w'\}$. Similarly $s_H(v'') = s_2 \setminus \{w''\}$.

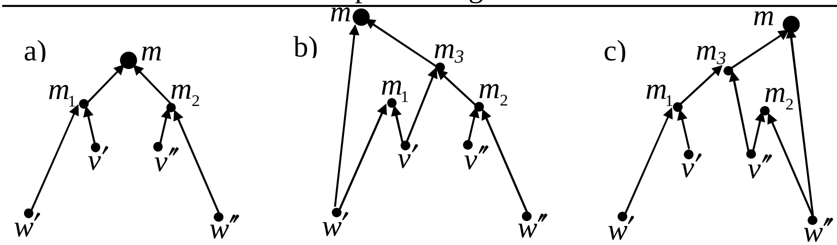

Figure 33. 2-hierarchy Resubordination for Strongly Narrowing Cost Function

If for groups s_1 and s_2 condition a) of Definition 11 holds then the hierarchy can be reconstructed into the hierarchy shown in Figure 33b). So, we can hire manager m_3 and immediately subordinate manager m_2 and employee v' to manager m_3. After that we can immediately subordinate worker w' and manager m_3 to manager m. Before the reconstruction manager's m cost equals $c(s_1,s_2)$ (the left-hand member of inequality a) of Definition 11). After the reconstruction the sum of managers' m_3 and m costs equals $c(s_1 \setminus \{w'\}, s_2) + c((s_1 \setminus \{w'\}) \cup s_2, \{w'\})$ (the right-hand member of inequality a) of Definition 11). Other managers' costs do not change.

Thus, condition a) of Definition 11 allows to immediately subordinate worker w' to the manager m with no hierarchy cost increase. Similarly if condition b) of Definition 11 holds then we can immediately subordinate worker w'' to the manager m with no hierarchy cost increase (see Figure 33c)).

Proposition 9. *If sectional cost function is strongly narrowing then there exists optimal consecutive hierarchy*.

Proposition 9 can be proven using the described above reconstructions of optimal 2-hierarchy (this hierarchy exists because of Proposition 7) till optimal consecutive hierarchy is constructed.

Proposition 9 leads to the fact that if for narrowing cost function inequality of Definition 11 hold then it is enough to obtain consecutive hierarchy with minimal cost to solve optimal hierarchy problem. As cited above consecutive hierarchy with minimal cost can be found analytically (see examples in Section 3.5) or using algorithms.

The class of strongly narrowing functions is less than the class of narrowing functions. However, many cost functions are strongly narrowing (see examples in Section 3.5).

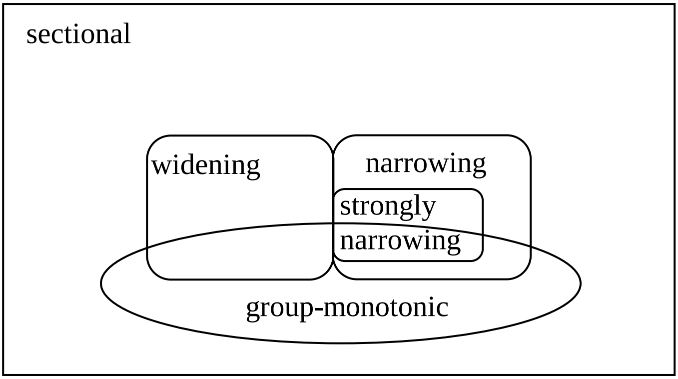

Figure 34. Interrelationship between Classes of Group-Monotonic, Strongly Narrowing, Narrowing and Widening Cost Functions

By definition the set of strongly narrowing functions is embedded into the set of narrowing functions. Examples of Section 3.5 show that there exist narrowing functions, which are not strongly narrowing. In extreme cases a sectional function may be both strongly narrowing and widening. Moreover, strongly narrowing function may be either group-monotonic or not. Interrelationship between classes of group-monotonic, strongly narrowing, narrowing and widening cost functions is shown in Figure 34.

In the following section we consider several examples of sectional cost functions and obtain optimal hierarchies using theoretical methods of Sections 3.2, 3.3 and 3.4.

3.5. Examples of Cost Function for Different Types of Interaction

Suppose for each worker $w \in N$ some *worker's complexity* $\mu(w)>0$ (positive real number) is given. Complexity may correspond with "work content" of the worker, his or her professional skills, etc. *Complexity of arbitrary group* of workers $s \subseteq N$ may be defined as total complexity of all workers in s: $\mu(s)=\sum_{w\in s}\mu(w)$. For example, complexity of the group may correspond with total "work content" of all workers in the group. Sectional cost function depends only on "quantity" of administrative labor (for example, planning and monitoring interactions between groups of workers controlled by immediate subordinates). So, manager's cost depends only on groups $s_1,\ldots,s_k$ controlled by all his or her immediate subordinates (see Section 3.1). Let's consider several examples of such sectional cost function that manager's cost depends only on complexities:

$$c(s_1,\ldots,s_k)=[\mu(s_1)^\alpha+\ldots+\mu(s_k)^\alpha-\max(\mu(s_1)^\alpha,\ldots,\mu(s_k)^\alpha)]^\beta, \quad \text{(I)}$$

$$c(s_1,\ldots,s_k)=[\mu(s_1)^\alpha+\ldots+\mu(s_k)^\alpha]^\beta, \quad \text{(II)}$$

$$c(s_1,\ldots,s_k)=[\mu(s)^\alpha/\max(\mu(s_1)^\alpha,...,\mu(s_k)^\alpha)-1]^\beta, \quad \text{(III)}$$

$$c(s_1,\ldots,s_k)=[\sum_{i=\overline{1,k}}(\mu(s)^\alpha-\mu(s_i)^\alpha)]^\beta, \quad \text{(IV)}$$

$$c(s_1,\ldots,s_k)=\mu(s)^\alpha/\min(\mu(s_1)^\beta,...,\mu(s_k)^\beta), \quad \text{(V)}$$

where $s=s_1\cup\ldots\cup s_k$ is the group controlled by the manager, $\mu(s_1),\ldots,\mu(s_k),\mu(s)$ are complexities of corresponding groups, $\alpha,\beta>0$ are some positive real numbers (parameters of the function).

Manager's cost functions (I)-(V) depend on complexities ("work content") of employees of the "*section*" (department, division or some over business unit) controlled by the manager immediately. Consider several meaningful interpretations of cost functions (I)-(V).

In different firms immediate subordinates (section) may be controlled using different mechanisms. Thus, interaction between the manager and his or her immediate subordinates (inside the section) may be organized in many ways. Below functions (I)-(V) will be interpreted as manager's cost for different ways of interaction of immediate subordinates inside the section. In management science literature many such ways are considered (see, for instance, Davies, Smith and Twigger (1991), Manz and Sims (1987), Peters (1987), Oldman and Hackman (1981), Jago and Vroom (1975)). Below we attempt to describe it mathematically.

Suppose there exists a "*semi-leader*" among immediate subordinates (inside the section). This semi-leader copes with his or her tasks completely even with no superiors' control (see, for instance, Jago and Vroom (1975)). Function (I) may correspond with this way of interaction. Manager's cost (I) depends on complexities of groups controlled by all immediate subordinates except the semi-leader. We mean that the immediate subordinate with maximal complexity (the maximal "work content", professional skills, etc.) is a semi-leader.

Suppose there *does not exist a* "*leader*" among manager's immediate subordinates. Thus, the manager spends some efforts to control each of his or her immediate subordinates. Therefore, manager's cost may depend on complexities of all groups controlled by immediate subordinates. Function (II) may correspond with this way of interaction.

Suppose there exists a "*leader*" among immediate subordinates (inside the section). The "leader" helps to solve problems of other immediate subordinates (for example, using his or her experience or authority). Therefore, the cost of immediate superior of the leader decreases (see, for instance, Jago and Vroom (1975)). Function (III) may correspond with this way of interaction. Manager's cost (III) depends on complexity of the whole group controlled by the manager and complexity of the group controlled by the leader, which is immediately subordinated to the manager. Among all immediate subordinates the leader controls the group with maximal complexity (for example, the leader may have maximal professional skills). The greater is this complexity, the greater is the importance of the leader among other immediate subordinates and the less is the cost of immediate superior. Therefore, in function (III) the complexity of the group controlled by the manager is divided by the complexity of the group controlled by immediately subordinated leader.

Function (IV) corresponds with cost of *individual interactions between the manager and all his or her immediate subordinates*. The cost depends on differences between complexity of the group controlled by the manager and complexities of groups controlled by immediate subordinates. Consider an example. A manager m controls group $s_H(m)$. In process of individual interaction with his or her immediate subordinate m_1 the manager m may inform m_1 about the part of the group $s_H(m)$, which is not controlled by m_1. The volume of this information may depend on difference of complexities $\mu(s_H(m))$ and $\mu(s_H(m_1))$. Manager's cost (IV) depends on the sum of such volumes of information for all immediate subordinates.

Suppose, among immediate subordinates (inside the section) there exists an employee, that controls the group with small complexity. This employee may have *little qualification*. Other immediate subordinates have greater qualification because they control more complex groups. Low-qualified immediate subordinate may increase manager's cost. To control this subordinate the manager may spend much effort. So, manager's cost may increase because he or she is diverted from solving more complex problems (just such problems must be solved by this manager). Function (V) may correspond with this way of interaction. Manager's cost (V) depends on complexity of the whole group controlled by the manager and complexity of the group controlled by the low-qualified employee, which is immediately subordinated to the manager. The less is the minimal qualification of subordinated employees the greater is the cost of immediate superior. Therefore, in function (V) the complexity of the group controlled by the manager is divided by the minimal complexity of the groups controlled by immediately subordinated employees.

So, functions (I)-(V) may correspond with managers' cost in real firms. Let's solve optimal hierarchy problem for these functions. For functions (I)-(IV) we use theoretical methods described in Sections 3.2, 3.3 and 3.4. For function (V) we use continuous approximation method (see Section 3.6).

Obviously functions (I) and (II) are group-monotonic and functions (III), (IV) and (V) are not group-monotonic. Let's examine narrowing, widening and strong narrowing conditions for these functions. To examine these conditions we use the following inequalities:

$$(x_1+...+x_k)^\gamma \geq x_1^\gamma+...+x_k^\gamma \text{ for any } x_1\geq 0,\dots,x_k\geq 0 \text{ and } \gamma\geq 1, \qquad (37)$$

$$(x_1+...+x_k)^\gamma \leq x_1^\gamma+...+x_k^\gamma \text{ for any } x_1\geq 0,\dots,x_k\geq 0 \text{ and } \gamma\leq 1. \qquad (38)$$

The inequalities (37) and (38) are particular cases of the Minkovski inequality (see, for instance, Hardy, Littlewood and Polya (1934)).

Proposition 10. *Function* (I) *is widening for* $\beta \leq 1$*, is narrowing for* $\beta \geq 1$*, is strongly narrowing for* $\beta \geq 1$ *and* $\alpha\beta \geq 1$.

To prove Proposition 10 it is enough to examine inequalities for narrowing and widening conditions (see inequalities (35) and (36) in Section 3.3), inequalities for strong narrowing condition (see Definition 11 in Section 3.4).

Proposition 10 allows to obtain optimal hierarchy for function (I). If $\beta \leq 1$ then two-tier hierarchy is optimal (see Proposition 8). If $\beta \geq 1$ then 2-tree with minimal cost is optimal (see corollary from Propositions 6 and 7). We can find this tree using algorithms described in Mishin and Voronin (2001). If $\beta \geq 1$ and $\alpha\beta \geq 1$ then consecutive hierarchy with minimal cost is optimal (see Proposition 9). Mishin and Voronin (2003) prove that a consecutive hierarchy with maximal complex worker in the first position (see Figure 32) has minimal cost (the order of other workers is unimportant). Figure 35 illustrates optimal hierarchies for function (I).

So, for cost function (I) Propositions 7, 8 and 9 allow to obtain optimal hierarchy analytically for all cases, except the parameter region $\beta > 1$ and $\alpha\beta < 1$. In this region optimal hierarchy problem is too much simplified (it is enough to obtain 2-tree with minimal cost) and there exist algorithms solving the problem.

Line $\beta = 1$ draws a distinction between narrowing and widening regions. If $\beta = 1$ then function (I) both narrowing and widening. So, two-tier hierarchy with single manager and some 2-tree with *n*–1 managers are optimal hierarchies. If β increases, then only 2-tree is optimal. If β decreases, then only two-tier hierarchy is optimal. The region $\beta = 1$, $\alpha \geq 1$ shows that cost function may be both widening and strongly narrowing.

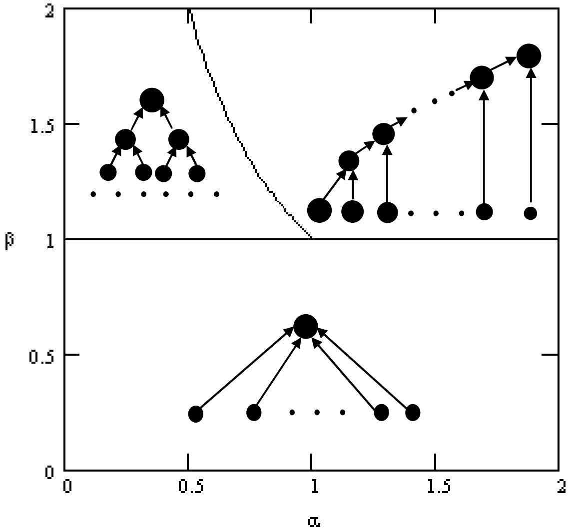

Figure 35. Forms of Optimal Hierarchy for Function (I)

Proposition 11. *Function* (II) *is widening for* $\beta \leq 1$, *is widening on non-overlapping groups for* $\beta > 1$ *and* $\alpha \geq 1$, *is neither widening nor narrowing for* $\beta > 1$ *and* $\alpha < 1$.

To prove Proposition 11 it is enough to examine inequalities for narrowing and widening conditions (see inequalities (35) and (36) in Section 3.3).

Thus, if $\beta \leq 1$ or $\beta > 1$ and $\alpha \geq 1$ then for function (II) two-tier hierarchy is optimal (see Proposition 8 and corollary). Figure 36 illustrates optimal hierarchies for function (II).

So, for cost function (II) Proposition 8 allows to obtain optimal hierarchy for all cases, except the parameter region $\beta > 1$ and $\alpha < 1$.

In the region $\beta > 1$ and $\alpha < 1$ Proposition 11 implies that function (II) is neither widening, nor narrowing even on non-overlapping groups. Therefore, for this region Proposition 7 and 8 can not help to obtain optimal hierarchy. However, function (II) is group-monotonic.

Thus a tree with minimal cost is optimal (see Proposition 6), we only need to obtain such tree.

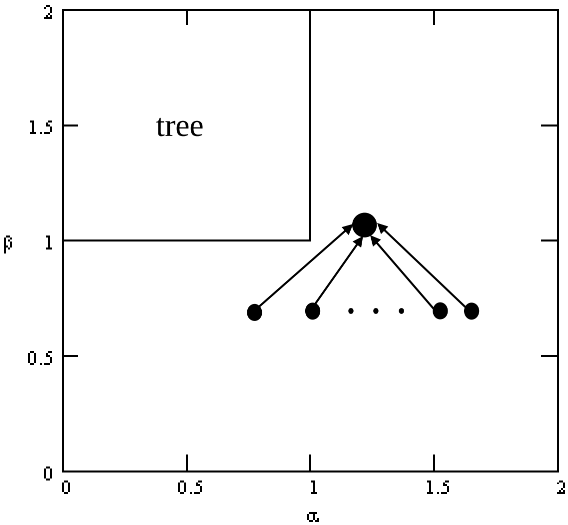

Figure 36. Forms of Optimal Hierarchy for Function (II)

In Section 3.6 we describe such analytical method that allows to obtain minimal cost tree for several cost functions. Moreover, for arbitrary sectional cost function we can obtain minimal cost tree using algorithms described in Mishin and Voronin (2001, 2003) (see brief description in Section 3.2). Let's illustrate algorithm's result for an example.

Consider seventy workers (*n*=70) with equal complexity (so all workers are identical for the cost function). Let's apply exact algorithm for function (II) with parameters $\alpha = 0.5$ and $\beta = 1.5$. For this case the optimal hierarchy is shown in Figure 37. The workers in the figure are denoted by numbers.

In the optimal hierarchy workers $w_1,\ldots,w_{40}$ are grouped in sections with four workers (four workers are subordinated to one manager on the second tier). Workers $w_{41},\ldots,w_{70}$ are grouped in sections with five workers. There are sixteen managers on the second tier. These managers are controlled by symmetric 4-tree (4 managers on the third tier and the single top manager). If $n = 4^3 = 64$ then symmetric 4-tree is optimal. In the considered case n=70 six "additional" workers are distributed between managers on the second tier with no modification of managers' subordination.

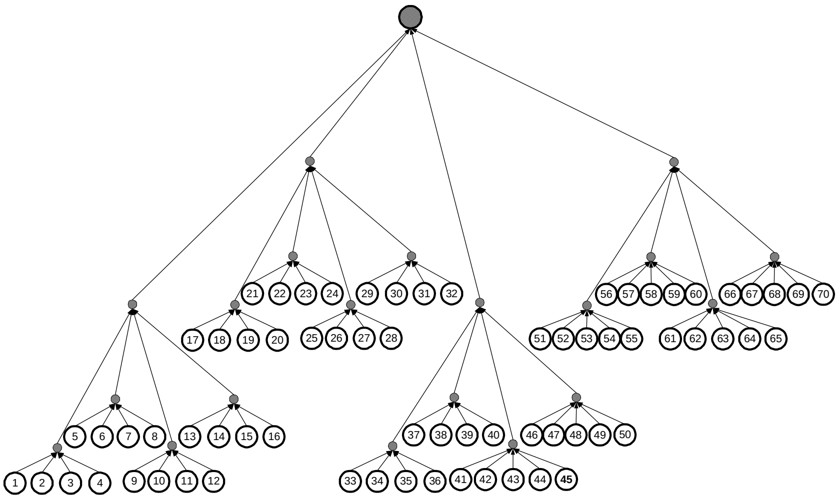

Figure 37. The Optimal Hierarchy for Function (II) with $\alpha = 0.5$ and $\beta = 1.5$

If all workers have equal complexity then in many cases optimal hierarchy looks like symmetric r-tree (for example, if n = 25, n = 125 or n = 625 then symmetric 5-tree is optimal[67]). If β equals to one then function (II) is widening and two-tier hierarchy becomes optimal ($r = +\infty$). If β increases, then 2-tree becomes optimal ($r = 2$). In the considered example 2-tree is optimal for $\beta \geq 3$.

Let's consider cost function (III).

Proposition 12. *Function* (III) *is strongly narrowing for* $\beta \geq 1$.

To prove Proposition 12 it is enough to examine inequalities for narrowing condition (see inequality (35) on page 88) and strongly narrowing condition (see Definition 11 on page 98).

[67] For n=125 and n=625 the tree is obtained using heuristic algorithm.

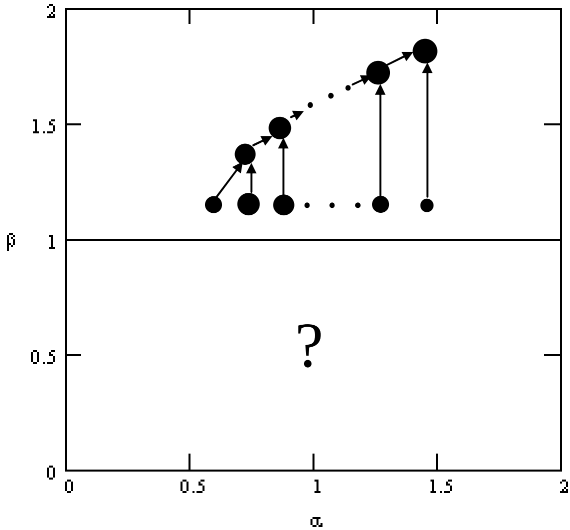

Figure 38. Form of Optimal Hierarchy for Function (III)

Proposition 12 allows to obtain optimal hierarchy for function (III) and $\beta \geq 1$. In this case consecutive hierarchy with minimal cost is optimal (see Proposition 9). Mishin and Voronin (2003) prove that consecutive hierarchy with the following property has minimal cost. The complexity does not increase from the worker in the second position (see Figure 32) to the worker in the last position. Thus, to solve optimal hierarchy problem it is enough to find the worker for the first position. Figure 38 illustrates optimal hierarchy for function (III).

So, for cost function (III) and $\beta \geq 1$ Proposition 9 allows to obtain optimal hierarchy analytically.

For $\beta < 1$ we can find the tree with minimal cost using algorithms. But this tree may be non-optimal because function (III) is not group-monotonic. By now methods to solve optimal hierarchy problem for function (III) and $\beta < 1$ are unknown.

Proposition 13. *Function* (IV) *is narrowing for* $\beta \geq 1$.

To prove Proposition 13 it is enough to examine inequality for narrowing condition (see inequality (35) on page 88).

Narrowing cost function may be not strongly narrowing. To prove it let's consider an example. Let n=4 and all employees have the same complexity $\mu(w_1) = \ldots = \mu(w_4) = 1$. Consider cost function (IV) with parameters $\alpha = 1$ and $\beta \geq 1$. If narrowing cost function (IV) is strongly narrowing, then the consecutive hierarchy with minimal cost is optimal (see Proposition 9). In the example all consecutive hierarchies have the same cost $2^\beta + 3^\beta + 4^\beta$ (see Figure 39a)).

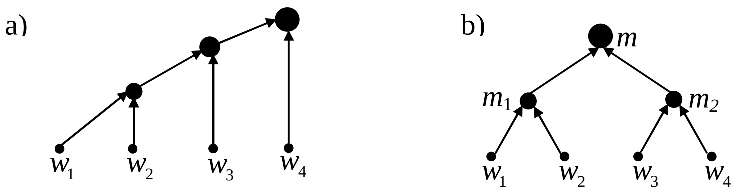

Figure 39. Non-Optimality of Consecutive Hierarchies for Cost Function (IV)

2-tree is shown in Figure 39b). The cost of the tree equals $2^\beta + 2^\beta + 4^\beta$. Thus, the cost of the 2-tree is less than the cost of a consecutive hierarchy. Therefore, consecutive hierarchy is not optimal. So, in the considered example narrowing cost function is not strongly narrowing

Proposition 13 allows to obtain optimal hierarchy for function (IV) and $\beta \geq 1$. In this case 2-hierarchy with minimal cost is optimal (see Proposition 7). Figure 40 illustrates optimal hierarchy for function (IV).

By now methods to solve optimal hierarchy problem for function (IV) and $\beta < 1$ are unknown. If $\beta \geq 1$, then Proposition 7 allows to simplify the problem (it is enough to obtain 2-hierarchy with minimal cost). By now it is unknown if 2-tree with minimal cost is optimal hierarchy or not. The tree with minimal cost may be obtained using algorithms solving this problem for arbitrary sectional function. But in some cases for function (IV) there exists much more efficient algorithm.

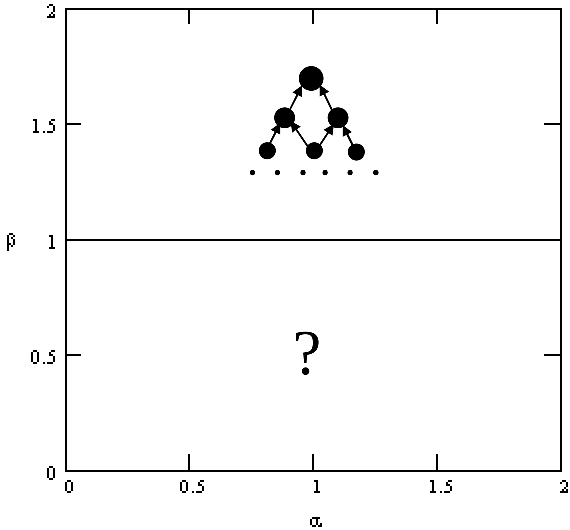

Figure 40. Form of Optimal Hierarchy for Function (IV)

Voronin and Mishin (2003) prove that for function (IV) with $\alpha = 1$ and $\beta = 1$ **the problem of searching out tree with minimal cost is equivalent to the problem of construction optimal alphabetic code**. This is well-known problem of discrete mathematics. There is given some alphabet with *n* symbols. The probability of each symbol appearance is also given. It is necessary to define such code word (several bits) for each symbol that any coded text could be uniquely decipher and expectation of length of code word would be minimal (so the expectation of coded text length would be minimal too). The workers may be interpreted as symbols and worker's complexities may be interpreted as probabilities of symbol appearance. In any 2-hierarchy there are two incoming edges for each manager (each node except the first tier). If we write zero in one edge and one in another edge then any path from the top manager (the top node) to the worker (corresponding with the symbol) defines the code word. For function (IV) with $\alpha = 1$ and $\beta = 1$ the cost of the 2-hierarchy equals to expectation of length of code word. Thus, 2-tree with minimal cost corresponds with optimal alphabetic code.

Therefore, for function (IV) with $\alpha = 1$ and $\beta = 1$ we can obtain 2-tree with minimal cost using Huffman algorithm (see Huffman (1952)). Two workers with minimal complexity are subordinated to one manager. Then this manager is considered instead of two subordinated workers and the algorithm continues similarly. As a result, we obtain 2-tree with minimal cost. Function (IV) is narrowing (see Proposition 13). Thus, the cost of the 2-tree obtained by the algorithm is minimal cost among all trees. Complexity of the algorithm equals $n \log n$.

For $\alpha = 1$ and $\beta = 1$ the example of minimal cost tree (obtained by Huffman algorithm) is shown in Figure 39b). In this example workers have the same complexity and the number of employees equals to the power of 2. Therefore, symmetric 2-tree has minimal cost. In this tree immediate subordinates of any manager control groups with the same complexity. In other cases the minimal cost tree may be non-symmetric. However, in any case Huffman algorithm "divides" the group controlled by a manager into two subgroups with "approximately equal" complexities. For example, in Figure 39b) manager *m* controls group $N=\{w_1,w_2,w_3,w_4\}$ with complexity 4. This complexity is "divided" into halves between managers m_1 and m_2 immediately subordinated to the manager *m*.

On the whole Section 3.5 shows that theoretical methods described in Sections 3.2, 3.3 and 3.4 allow to obtain optimal hierarchy for many cost functions. However, in several cases these methods can not help to solve optimal hierarchy problem. In the next section we describe continuous approximation method, which allows to obtain minimal cost tree for the so called homogeneous cost functions. Function (V) is analyzed using this method.

3.6. Continuous Approximation Method for Searching the Tree with Minimal Cost

In Section 3.3 we show that widening and narrowing functions imply optimality of two extreme hierarchies: two-tier hierarchy and 2-hierarchy. Usually in real firms there are some "intermediate" hierarchies with span of control $2 < r < +\infty$. Therefore, to model many real firms we have to examine neither widening nor narrowing cost functions. Thus, it is important to solve optimal hierarchy problem for this case. In this section we describe a method of searching out tree with minimal cost. If the cost function is group-monotonic then this tree is optimal (see Proposition 6). For other functions this is the best tree. Bellow we use continuous approximation method to examine cost function (V) (see Section 3.5).

Optimal hierarchy problem is discrete optimization problem. Therefore, it is difficult to solve it analytically. One possible way of solution is to consider corresponding continuous problem with continuum set of workers. The exploration of continuous problem of searching out minimal cost tree for sectional cost functions was pioneered by Goubko (2002). In some cases after the continuous problem is solved we can prove that corresponding tree minimizes cost for the original discrete problem.

Suppose we have to obtain minimal cost tree and cost function $c(s_1,\dots,s_k)$ depends only on complexities of groups $s_1,\dots,s_k$. Thus, the cost function is given by $c(\mu(s_1),...,\mu(s_k))$ (see, for example, functions (I)-(V) in Section 3.5) [68].

Consider only *homogeneous* cost functions satisfying the following condition. For any $y>0$ the equality $c(y\mu(s_1),...,y\mu(s_k)) =$ $= \varphi(y)c(\mu(s_1),...,\mu(s_k))$ holds, where $\varphi(\cdot)$ is some continuously increasing function. It can be proven (Goubko (2002)) that $\varphi(y) = y^{\gamma}$, where γ is *homogeneity coefficient*. If a cost function is homogeneous, then scale of complexity is of no importance. If we multiply all workers' complexities by the same multiplier y, then costs of all hierarchies are multiplied by y^{γ}. Therefore, scale of complexity does not affect on optimality of hierarchies.

Let's define continuous problem corresponding with the discrete problem.

Let $x = \mu(w_1) + \ldots + \mu(w_n)$ be total complexity of workers in the discrete problem. Suppose in the continuous problem the set of workers equals to the segment $N=[0;x]$. An individual worker is a point of this

[68] For any tree the groups $s_1,\dots,s_k$ are non-overlapping. So, $\mu(s_1 \cup \ldots \cup s_k) = \mu(s_1) + ... + \mu(s_k)$ and we may suppose that functions (I)-(V) depend only on $\mu(s_1),...,\mu(s_k)$.

segment. The top manager m controls all segment N (all workers). The segment is divided into parts among managers $m_1,\dots, m_k$ immediately subordinated to the top manager. Each of the managers $m_1,\dots, m_k$ controls some part of the segment N. Thus, the segment N is divided into smaller segments with lengths $x_1,\dots,x_k>0$ controlled by managers $m_1,\dots, m_k$ correspondingly, $x_1+\dots+x_k=x$. The segment with length x_i controlled by the manager m_i is divided into smaller segments controlled by his or her immediate subordinates, $1 \le i \le k$. These segments are divided again, etc. The tree infinitely "grows". In the tree each manager corresponds with a segment. The length of the segment equals to complexity of the group subordinated to the manager. If manager's immediate subordinates control segments with lengths $x_1,\dots,x_k$ then manager's cost equals $c(x_1,\dots,x_k)$. Cost of a tree equals to total cost of all managers in the tree. *It is necessary to obtain infinite tree with minimal cost.*

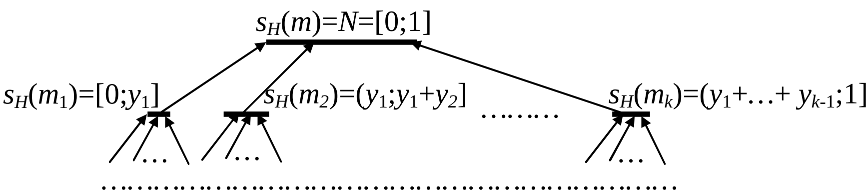

Figure 41. The Top Piece of Self-Similarly Tree with Proportion $y_1,\dots,y_k$ and $x=1$

Goubko (2002) proves that **for any homogeneous cost function there exists self-similarly tree H with minimal cost**. In H each segment is divided in the same proportion $y_1,\dots,y_k>0$ regardless of hierarchical tier, $y_1+\dots+y_k=1$. The top piece of self-similarly tree is shown in Figure 41. Controlled segments are shown instead of managers. Immediate subordinates $m_1,\dots,m_k$ of the manager m control segments with lengths $y_1x,\dots,y_kx$. Therefore, manager's m cost equals $x^\gamma c(y_1,\dots,y_k)$. Total cost of managers $m_1,\dots,m_k$ equals $x^\gamma c(y_1,\dots,y_k)(y_1^\gamma+\dots+ y_k^\gamma)$. Expression in the brackets squares for the managers of the next tier, cubes for the manager of the next tier, etc. For $\gamma>1$ such expressions are geometric series with multiplier $y_1^\gamma+\dots+ y_k^\gamma<1$ (this inequality follows from inequality (37) on page 103 because $y_1+\ldots+y_k=1$). Thus, the cost of self-similarly tree H equals to the sum of infinitely decreasing geometric series:

$$c(H)=x^{\gamma}c(y_1,\ldots,y_k)/(1-\sum_{i=\overline{1,k}}y_i^{\gamma}). \qquad (39)$$

One of such trees minimizes cost. So, it is enough to find $k\geq 2$ and proportion $y_1,\ldots,y_k$ minimizing expression (39). Corresponding tree is desired infinite tree with minimal cost.

Let's obtain tree with minimal cost for function (V). In any tree immediate subordinates of common manager control non-overlapping groups (segments). For any non-overlapping groups $s_1,\ldots,s_k$ equality $\mu(s_1\cup\ldots\cup s_k)=\mu(s_1)+\ldots+\mu(s_k)$ holds. Therefore, function (V) is given by:

$$c(\mu(s_1),\ldots,\mu(s_k))=(\mu(s_1)+\ldots+\mu(s_k))^{\alpha}/\min(\mu(s_1)^{\beta},\ldots,\mu(s_k)^{\beta}). \qquad (40)$$

Expression (40) implies that function (V) is homogeneous. Homogeneity coefficient γ equals $\alpha-\beta$. Thus, we can minimize the cost (39) and obtain infinite tree with minimal cost.

Proposition 14. *Let r_* denote one of two integer numbers closest to the value $r_0=((\alpha-1)/\beta)^{1/(\alpha-\beta-1)}$. For continuous problem with cost function* (V) *and* $\alpha-\beta>1$ *symmetric r_*-tree minimizes cost. In this tree any manager has exactly r_* immediate subordinates controlling groups with equal complexity.*

In the proof of Proposition 14 we show that for function (V) values $y_1=\ldots=y_k=1/k$ minimize expression (39). So, symmetric tree minimizes cost. Thus, it is enough to find k minimizing expression (39). The minimum point $r_0=((\alpha-1)/\beta)^{1/(\alpha-\beta-1)}$ may be non-integer value. Therefore, r_* is maximal integer less than or equal to r_0 or r_* is minimal integer greater than or equal to r_0 (to define r_* it is enough to substitute these two values in expressions (39) and (40)).

For function (V) with $\alpha-\beta>1$ Proposition 14 solves the continuous problem. Consider corresponding discrete problem with number of workers $n=r_*^j$ (n is some power of r_*) and the same workers' complexities $\mu(w_1)=\ldots=\mu(w_n)=1/n$. In this case top j tiers of the infinite symmetric r_*-tree are just discrete tree controlling workers $w_1,\ldots,w_n$

(these workers correspond with the tier j+1). And cost of this part of the infinite tree equals to cost of discrete tree. Therefore, for $\boldsymbol{n}= r_*^j$ **and workers with the same complexity symmetric *r*∗-tree minimizes cost for the discrete problem**[69]. Thus, in this case we solve the discrete problem using continuous approximation method. The solution may be shown using the diagram.

In Figure 42 the line β=α–1 is shown. The region below this line is divided into regions with the same r_*. In each of these regions optimal span of control does not change. In the top right region symmetric 2-tree minimizes cost[70]. In the region below symmetric 3-tree minimizes cost. In the next region symmetric 4-tree minimizes cost, etc. If parameters tend to the point (1;0) then r_* grows infinitely (for r_*<10 in the figure regions are denoted by numbers). If α increases then in Figure 42 curves exponentially decrease. In Figure 42 2-tree and 3-tree are shown. In these trees the group controlled by a manager is “divided” into sub-groups with the same complexity among manager’s subordinates. Trees for more r_* may be shown similarly.

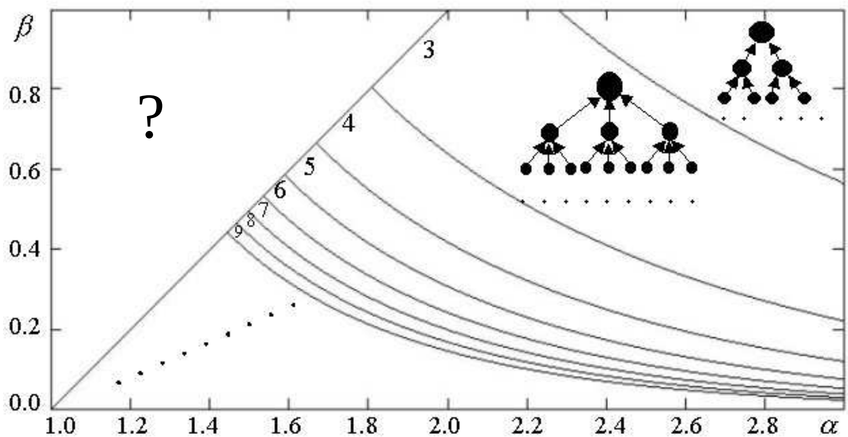

Figure 42. Minimal Cost Trees for Function (V)

[69] Otherwise we can reduce the cost of the infinite tree using the discrete tree with less cost to construct top j tiers of the infinite tree.

[70] Also this tree minimizes cost for sufficiently large α, β in any line β=b(α–1), 0<b<1.

Parameter β may be interpreted as degree of unfavorable influence of little qualification. If β tends to zero then we can subordinate low-qualified employees (controlling groups with low complexity) to the manager with no his or her cost sufficiently increase (see expression (40)). Therefore, if β tends to zero then optimal span of control r_* tends to $+\infty$. Thus, for sufficiently small β two-tier hierarchy with single manager minimizes cost (if β=0 then function (V) is widening and two-tier hierarchy is optimal for any number of workers).

There exists the limit of the value r_0 (see Proposition 14) by parameters tending to the critical line β=α–1. This limit equals $e^{1/\beta}$. So, parameter regions with fixed r_* "reach" the critical line.

For special cost function Qian (1994) also considers the problem of searching out minimal cost tree. If real number of immediate subordinates are possible, then Qian (1994) proves that optimal span of control equals e (each manager has e immediate subordinates). This result coincides with the result for function (V) with α=2 and β=1 ($\lim r_0 = e^{1/\beta} = e$).

Figure 42 shows that for any $r \geq 2$ there exists such region of parameters α and β that symmetric r-tree has minimal cost. In many real firms span of control ranges from several immediate subordinates to hundreds immediate subordinates (Mintzberg (1979)). The values $2 < r < +\infty$ may be interesting to model such firms.

The method described in this section allows to obtain tree with minimal cost analytically for neither widening nor narrowing homogeneous cost functions. If such cost function is group-monotonic, then the obtained tree is optimal.

3.7. Optimal Hierarchy Controlling Several Groups of Workers

Definition 1 implies that in any hierarchy there exists the manager controlling all workers. Proposition 1 implies that there exists optimal hierarchy with single manager controlling all other employees. So, there exists the single top manager with authority over all employees.

Definition 1 is quite reasonable if a hierarchy must control all workers' interactions. But the following more complex problem may be considered too. Suppose there is some technology of l goods production (for example, there is some technological network between the workers). Not all workers but only some of them may produce each good according to the technology. So, for the i-th good there exists some group of workers s_i who produce this good. In some cases there is no need for one manager controlling all workers. To produce the i-th good it is enough to control all interactions in the group s_i. Therefore, managers must control interactions in some given groups $s_1,\ldots,s_l$.

Consider the following example. We have to produce two goods. Workers w_1 and w_2 supply all firm by raw materials. Workers w_7 and w_8 sell all produced goods. Workers w_3 and w_4 produce the first good. And workers w_5 and w_6 produce the second good. Suppose there are no interactions between workers producing different goods. To supply raw materials, produce and sell the first good it is necessary to control interactions inside the group $s_1=\{w_1,w_2,w_3,w_4,w_7,w_8\}$. Similarly to supply raw materials, produce and sell the second good it is necessary to control interactions inside the group $s_2=\{w_1,w_2,w_5,w_6,w_7,w_8\}$.

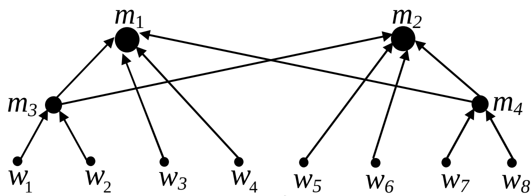

Figure 43. An Example of Hierarchy Controlling Two Groups of Workers

If there is no need for one manager controlling all workers then the hierarchy in Figure 43 controls production of two goods. In the hierarchy the head of the supply department (manager m_3) and the head of the sales department (manager m_4) take part in both goods production. These managers are immediately subordinated to managers m_1 and m_2, who control all employees taking part in the output of the first and the second goods respectively. m_1 and m_2 immediately control workers producing corresponding goods. Thus, we can consider the following formal definition.

Definition 12. *A directed graph* $H=(N\cup M,E)$ *with a set of subordination edges* $E\subseteq(N\cup M)\times M$ *is called the hierarchy control-*

ling given groups of workers $s_1,\dots,s_l$ if H is acyclic, any manager has at least one subordinated employee and for each group from $s_1,\dots,s_l$ there exists a manager controlling this group in H. Let $\Omega(s_1,\dots,s_l)$ *be the set of all such hierarchies.*

Managers' costs and costs of hierarchies from the set $\Omega(s_1,\dots,s_l)$ may be defined using a sectional cost function (see Definition 7 on page 79) just as costs of hierarchies from the set $\Omega(N)$. So, we can consider the problem of searching out some optimal hierarchy, which has minimal cost among all hierarchies from $\Omega(s_1,\dots,s_l)$.

If a hierarchy must contain a manager controlling all workers, then Definition 12 may be interpreted too. In this case we can add the group $s_{l+1}=N$ to the groups $s_1,\dots,s_l$. Then any hierarchy from $\Omega(s_1,\dots,s_l,s_{l+1})$ satisfies all conditions of Definition 1 (the hierarchy contains the manager controlling all workers). **In this case managers controlling groups $s_1,\dots,s_l$ may correspond with heads of some sections** (departments, divisions, etc.), **that must be organized**.

In the example considered above (see Figure 43) it may be necessary that in any hierarchy there are managers controlling all employees taking part in the output of each of goods and heads of the supply and sales departments. Thus, in any hierarchy there exist managers controlling the following groups: $s_1=\{w_1,w_2,w_3,w_4,w_7,w_8\}$, $s_2=\{w_1,w_2,w_5,w_6,w_7,w_8\}$, $s_3=\{w_1,w_2\}$, $s_4=\{w_7,w_8\}$. In this case we can consider a set of hierarchies $\Omega(s_1,\dots,s_4)$ or a set $\Omega(s_1,\dots,s_4,N)$ – if a hierarchy must contain a manager controlling all workers.

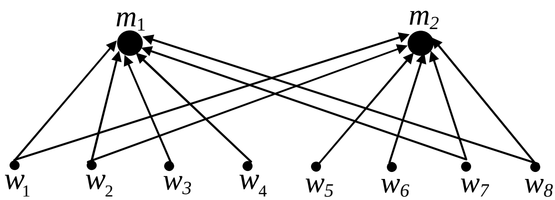

Figure 44. Two-Tier Hierarchy Controlling Two Groups of Workers

Set $\Omega(s_1,\dots,s_l)$ is too large regardless of the fact that Definition 12 applies restrictions. Therefore, it is very difficult to find optimal hierarchy using enumerative technique. So, it is necessary to develop methods, which under some restrictions help to search out an optimal hierarchy controlling given groups $s_1,\ldots,s_l$.

Also a hierarchy controlling groups $s_1,\ldots,s_l$ may be interpreted in the following way. In Section 3.4 we briefly describe information processing model (Marschak and Radner (1972)). Some information incomes from n sources corresponding with workers. The managers have to process information and obtain common control action for the whole firm (have to calculate some function). An associative function is considered (for example, addition). So, the function value does not depend on order of calculations. In all known models the calculation of singe function depending on all n sources (variables) is considered (see, for instance, Keren and Levhari (1983, 1989), Radner (1993), Van Zandt (1996)). But in real firm it may be necessary to calculate several control actions (several functions depending on different parts of variables). For example, let the group s_1 correspond with some workshop. Then managers have to gather information from workers of the workshop and calculate some control action for the workshop s_1 only. Similarly it is necessary to calculate control actions for workshops $s_2,\ldots,s_l$. Only a hierarchy controlling groups $s_1,\ldots,s_l$ calculates all control actions. Optimal hierarchy minimizes calculation cost. By now methods of searching out optimal hierarchy calculating several functions are unknown (Radner (1992)). Therefore, even such methods for very special cases may be interesting.

If groups $s_1,\ldots,s_l$ are non-overlapping then the problem of searching out optimal hierarchy controlling groups $s_1,\ldots,s_l$ decomposes to l independent optimal hierarchy problems. Managers controlling groups s_i and s_j for $i \neq j$ have no common subordinates because in this case groups s_i and s_j have common workers. So, a hierarchy from set $\Omega(s_1,\ldots,s_l)$ decomposes to l independent hierarchies from sets $\Omega(s_1),\ldots,\Omega(s_l)$. In this case it is enough to obtain l optimal hierarchies controlling one group. So, the problem is completely reduced to the problem considered above.

If groups $s_1,\ldots,s_l$ overlap, then the problem is much more complicated. For example, in Figure 43 managers m_1 and m_2 control groups $s_1=\{w_1,w_2,w_3,w_4,w_7,w_8\}$ and $s_2=\{w_1,w_2,w_5,w_6,w_7,w_8\}$ respectively.

Groups overlap, $s = s_1 \cap s_2 = \{w_1, w_2, w_7, w_8\}$. We can hire a manager controlling the group $s = s_1 \cap s_2$ and immediately subordinate him or her to managers m_1 and m_2. Or we can hire managers controlling parts of the group s and immediately subordinate them to managers m_1 and m_2 (for example, managers m_3 and m_4 in Figure 43). Managers m_1 and m_2 may control subordinated group independently (using subordinated managers or immediately, see Figure 44). Also the set $\Omega(s_1, s_2)$ contains many other hierarchies. So, when we construct the hierarchy controlling single group s_1 we must keep in mind that some managers may be used to construct the hierarchy controlling the group s_2 if it decreases the cost of total hierarchy. For arbitrary groups $s_1,\dots,s_l$ intersection structure may be too complex. Therefore, the problem is very complicated. Managers controlling the subgroups of $s_1 \cap \dots \cap s_l$ may be subordinated to all l managers controlling groups $s_1,\dots,s_l$. Similarly **it is necessary to analyze every intersection of groups $s_1,\dots,s_l$** (generally there are $2^l - 1$ such intersections).

However, regardless of problem's complexity some results described above can be generalized for the problem of searching out optimal hierarchy controlling several groups.

Proposition 15. *If sectional cost function is narrowing then there exists optimal 2-hierarchy $H \in \Omega(s_1, \dots, s_l)$ controlling groups $s_1,\dots,s_l$.*

Proposition 15 can be proven in the same way as corresponding proposition for hierarchies controlling one group (see Proposition 7 on page 89). To prove the proposition we can reconstruct an optimal hierarchy in the following way. If any manager m has three or more immediate subordinates $v_1,\dots,v_k$, then we can hire new immediately subordinated manager m' controlling two or more employees from $v_1,\dots,v_k$. After that manager m immediately controls other employees from $v_1,\dots,v_k$ and manager m'. Narrowing cost function implies that such reconstruction does not increase cost. Therefore, the reconstructed hierarchy is optimal.

For several groups we can similarly reconstruct optimal hierarchy because we only hire new managers and do not eliminate managers (these managers may be necessary for several top managers controlling different groups). Converse proposition (two-tier hierarchy is optimal for widening cost function, see Proposition 8) is incorrect for several groups. For example, consider Figure 43 and widening cost function. If workers w_1 and w_2 are immediately subordinated to manager m_1, then manager's m_1 cost is less than total cost of managers m_1 and m_3 in Figure 43. But manager m_3 is necessary for both manager m_1 and manager m_2. Therefore, elimination of manager m_3 can increase the cost of the hierarchy even for widening cost function.

For narrowing cost functions Proposition 15 simplifies the problem of searching out optimal hierarchy controlling several given groups. In this case it is enough to consider only hierarchies with each manager having two immediate subordinates. Strong narrowing condition (see Definition 11 on page 98) allows to simplify the problem even more.

Proposition 16. *If sectional cost function is strongly narrowing then there exists optimal consecutive hierarchy* $H \in \Omega(s_1, \ldots, s_l)$ *controlling groups* $s_1, \ldots, s_l$.

Proposition 16 can be proven in the same way as corresponding proposition for hierarchies controlling one group (see Proposition 9 on page 99). The proof is based on reconstructions with no elimination of managers (only new managers are hired). Therefore, the proof of Proposition 9 is correct for several groups.

For strongly narrowing cost function Proposition 16 implies that consecutive hierarchy with minimal cost is optimal hierarchy controlling several groups of workers. Thus, it is enough to consider only hierarchies with each manager having one immediately subordinated worker and other immediately subordinated employee.

Mishin and Voronin (2002b, 2003) introduce **algorithm of searching out minimal cost consecutive hierarchy controlling several given groups of workers**. For arbitrary sectional function the complexity of the algorithm grows like $n2^n3^l$. Thus, complexity grows exponentially by number of workers n and by number of groups l. Testing of the algorithm shows that average complexity is lower for small n and l: the algorithm solves the problem for n and l less than 10-20.[71] Usually the number of groups l is not large (for example, usually the number of workshops, that must be organized, is less than

[71] Using personal computer in several minutes.

10). The restriction by n is much more important because large firm may contain hundreds or thousands workers. Consider particular case with sufficiently decreased complexity of the algorithm.

Suppose all workers are identical. Thus, we consider the cost function given by expression $c(|s_1|,\ldots,|s_k|)$. So, manager's cost depends only on the span of control k and on the numbers $|s_1|,\ldots,|s_k|$ of workers in the groups controlled by immediate subordinates (but not on individual workers in these groups!). For example, functions (I)-(V) (see Section 3.5) are given by $c(|s_1|,\ldots,|s_k|)$ if complexities of all workers are identical. This case is important because often workers may be considered as identical from the point of view of managers' costs. For identical workers Mishin and Voronin (2002b, 2003) introduce modification of the algorithm of searching out minimal cost consecutive hierarchy controlling several groups of workers. The complexity of the modification depends only on number of groups l (does not depend on number of workers n). The complexity of the modified algorithm grows like $2^{2l+1}3^l$. For identical workers the modified algorithm obtains the minimal cost consecutive hierarchy if the number of groups l is less than 10-20 regardless of the number of workers n.

If the cost function is strongly narrowing then algorithms obtain optimal hierarchy controlling several given groups of workers.

On the whole Chapter 3 shows that it is possible to explain sectional cost functions analytically. Regardless of the fact that optimal hierarchy problem is too complicated, in some cases it is solved (optimal hierarchy type is obtained). The methods solving the problem may be used for wide classes of sectional functions. Thus, problems of different fields of application may be solved using the same theoretical methods. Therefore, we can mathematically explore hierarchies in many firms.

Brief Summary and Concluding Remarks

The study of hierarchies helps to solve various practical management problems in firms. In management science literature numerous papers focus their attention on organizational hierarchies. By now too many empiric facts are gathered. These facts allow us to make different hypotheses about relationships between the type of the optimal hierarchy and the kind of business, parameters of environment, the size of the firm, the "age" of the firm, etc. (see, for instance, Mintzberg (1979)). Therefore, it is important to construct mathematical models, which are able to examine and systematize these facts and hypotheses.

In some papers optimal hierarchy problem is solved jointly with construction of control mechanisms. To obtain the solution of such a joint problem it is necessary to introduce certain assumptions (e.g. any hierarchy is a tree, only employees on adjacent tiers may interact directly, employees on one tier are identical, etc.). In this paper we dispense these stringent assumptions, but we do not consider control mechanisms. The proposed approach allows to construct the theoretical methods and to solve optimal hierarchy problem for a comparatively general framework. These methods can be used to solve many problems that have numerous applications in economics. Particularly, we model various effects occurring in real firms: relationships between the type of the optimal hierarchy and environment instability, standardization, the intensity of technological flows, horizontal and vertical integration, etc.

So, one can model many empirical relationships using sectional cost functions[72] introduced in this paper. Moreover, the class of sectional functions can be analyzed analytically and the optimal hierarchy can be found in several cases. Therefore, the sectional cost function appears to be a useful compromise between detailed description of the real firms and possibility of mathematical modeling. Thus, further development of the methods of the optimal hierarchy search for sectional cost functions seems perspective, among the following other general directions of future research.

1. Mechanism design. It is important to construct control mechanisms that minimize total wage of employees, which equals to the cost

[72] Manager's cost depends only on sets of workers controlled by employees immediately subordinated to the manager.

of the optimal hierarchy (this is minimal possible cost). Particularly, appropriate incentive mechanisms can be useful.

Mishin (2004a) constructs such mechanism in a complete information framework. This mechanizm provides minimal total wage by compensating the managers' costs. For the case of incomplete information it is necessary to take the "worst case" into consideration. For example, it may be necessary to compensate maximal total cost of all managers (this maximal cost depends on information available for some metacenter, for instance, the owner of the firm). However, in some cases excess incentives provide stability with respect to cost increase. If managers' cost increase, then a manager can restructure the subordinated part of the hierarchy with no assistance (at the expense of manager's own resources). It allows to "adapt" the firm to the cost modifications. Moreover upper tiers are the most stable (see Mishin (2004a)).

2. Dynamical models of the optimal hierarchy. Parameters of the cost function, the number of workers, certain workers, interaction schemes (e.g. technological network) can change with time. Therefore, the initially optimal hierarchy can later become non-optimal. However, the reconstruction of the hierarchy is associated with large cost. So, in dynamical models one has to compromise the total cost of all managers and the reconstruction cost. Thus, in the dynamical model the hierarchy with little reconstruction cost may be optimal even if total managers' cost is not minimal. Mishin (2002b) introduces a metric on the set of hierarchies. This metric is one of possible ways to define mathematically the reconstruction (restructuring) cost. Using this metric it is possible to model the restructuring effects numerically (Mishin (2002a, 2003a), Mishin and Voronin (2002a)). However, analytical methods for solving the dynamical problem of the optimal hierarchy are unknown so far.

Ideally the development of mathematical models should help to construct effective organizational hierarchies in real firms. In modern economy this problem is very important. We hope this paper will be useful for its solution.

Appendix (mathematical proofs)

Proof of Lemma 1. v is subordinated to m. So, any worker $w \in s_H(v)$ is subordinated to m because the path from w to v can be extended up to the path from w to m. That is $w \in s_H(m)$. Therefore, $s_H(v) \subseteq s_H(m)$.

If $w \in s_H(m)$ then there exists the path from w to m. This path contains the node v_j for some $1 \le j \le k$ as $(v_1,m),\ldots,(v_k,m)$ are the only edges incoming to m. So, $w \in s_H(v_j)$. Therefore, $s_H(m) \subseteq s_H(v_1) \cup \ldots \cup s_H(v_k)$. $s_H(v_j) \subseteq s_H(m)$ as v_j is subordinated to m for each $1 \le j \le k$. Thus, the equality $s_H(m) = s_H(v_1) \cup \ldots \cup s_H(v_k)$ holds. ■

Proof of Lemma 2. Let H be a tree. Assume $s_H(v_1) \cap s_H(v_2) \ne \varnothing$ for some manager m and two of his or her immediate subordinates v_1 and v_2. Then there exists a worker $w \in s_H(v_1) \cap s_H(v_2)$. The worker w is subordinated to the employees v_1 and v_2. So, there are two different paths from w to m (the first path contains the node v_1 and the second path contains the node v_2). These paths diverge at some node $v \in N \cup M$. Thus, the employee v has more than one immediate superior. It contradicts Definition 2. Thus, in the tree H any manager's immediate subordinates control non-overlapping groups of workers.

Let's prove converse proposition using the method of induction by number of workers n. Let any manager's immediate subordinates control non-overlapping groups of workers in the hierarchy H. Let m be the manager without superiors. By conditions of lemma m is the only manager without superiors.

When $n = |N| = 1$ all managers control the same group containing single worker. If some employee has two immediate superiors then there exist two different paths from this employee to m. These paths converge at some manager. So, this manager has two immediate subordinates controlling the same group. It contradicts the condition above.

Therefore, each employee but m has the single immediate superior. Thus, the hierarchy H is a tree.

Suppose the converse proposition is true for each $n < l$ for some number $l \geq 2$. Let $|N| = l$. The equality $s_H(m)=N$ holds because there exists manager controlling all workers and all managers are subordinated to m. If m has single immediate subordinate m' then m' controls the group N too. The manager m' may have single immediate subordinate too. So, we can construct such path with single subordination down to the managers m'' having two or more immediate subordinates $v_1,\ldots,v_k$, $k \geq 2$.

The equality $s_H(v_i) \cap s_H(v_j) = \varnothing$ is true for each $i \neq j$. Moreover, Lemma 1 leads to the fact that the equality $N = s_H(v_1) \cup \ldots \cup s_H(v_k)$ holds. So, $|s_H(v_i)| < |N| = l$ for each $1 \leq i \leq k$. Let H_i be the subgraph of the hierarchy H consisting of v_i and all his or her subordinates. Let v' be an employee of the hierarchy H_i and v'' be an employee of the hierarchy H_j for some $i \neq j$. Then $v' \neq v''$ and v' is not subordinated to v'' in H (otherwise $s_H(v') \subseteq s_H(v'') \subseteq s_H(v_j)$, but it is impossible because $s_H(v') \subseteq s_H(v_i)$ and $s_H(v_i) \cap s_H(v_j) = \varnothing$). So, the graphs $H_1,\ldots,H_k$ have no common nodes and no edges from one graph to another. In H_i single employee v_i has no superiors. Therefore, induction hypothesis leads to the fact that H_i is a tree controlling the workers from the group $s_H(v_i)$ and v_i is the root of this tree.

With the exception of m'' and his or her superiors each employee v of the hierarchy H is an employee of the hierarchy H_i for some $1 \leq i \leq k$ because v is subordinated to m and m''. So, the hierarchy H consists of k independent trees and there exist edges from the roots of these trees to m''. And H can contain the path from m'' to m with single subordination. Thus, H is a tree and the lemma is proven. ■

Proof of Lemma 3. Consider the set of two workers subordinated to the manager m: $\{w', w''\} \subseteq s_H(m)$.

Let $\{w', w''\} \subseteq s_H(v_j)$ for some $1 \leq j \leq k$. Then the workers w' and w'' are subordinated to v_j. So, the flow $f(w', w'')$ is not part of the internal flow controlled by the manager m.

Let $\{w', w''\} \not\subset s_H(v_j)$ for each $1 \le j \le k$. Suppose w' and w'' are controlled by some subordinate m' of the manager m ($\{w', w''\} \subseteq s_H(m')$). Then m' is not immediately subordinated to the manager m. Therefore, m' is subordinated to v_j for some j (the path from m' to m contains one of the immediate subordinates of the manager m). Lemma 1 implies that $\{w', w''\} \subseteq s_H(m') \subseteq s_H(v_j)$. It contradicts the above assumption. So, both w' and w'' are controlled by none of the subordinates of the manager m. Then the flow $f(w', w'')$ is part of the internal flow controlled by the manager m.

Thus, the sum $\sum\limits_{\substack{\{w',w''\} \subseteq s_H(m), \\ \{w',w''\} \not\subset s_H(v_j) \text{ для всех } 1 \le j \le k}} f(w', w'')$ contains the internal flows of the manager m and only such flows. ■

Proof of Lemma 4. Lemma 1 implies that $s_H(m_1) = s_H(v_1) \cup \ldots \cup s_H(v_k)$. We can remove the group $s_H(v_i)$ from this equality with no modification of the group $s_H(m_1)$ because $s_H(v_i) \subseteq s_H(v_j)$. Thus, the external flow $F_H^{ext}(m)$ does not change. In the expression for the internal flow $F_H^{\text{int}}(m_1)$ (see Lemma 3) the flows for all $\{w', w''\} \subseteq s_H(m_1)$, $\{w', w''\} \not\subset s_H(v_1), \ldots, \{w', w''\} \not\subset s_H(v_k)$ are summarized. The condition $\{w', w''\} \not\subset s_H(v_j)$ is sufficient for $\{w', w''\} \not\subset s_H(m_i)$. So, if we remove the group $s_H(v_1)$ (the condition $\{w', w''\} \not\subset s_H(v_1)$ is removed) then the internal flow $F_H^{\text{int}}(m)$ does not change. The flow of the manager m also does not change. Therefore, the equality $c(s_H(v_2), \ldots, s_H(v_k)) = c(s_H(v_1), \ldots, s_H(v_k))$ holds. Thus, the inequality in the lemma statement holds. In the basic model the inequality holds as equality. In other cases the inequality may hold strictly. ■

Proof of Proposition 1: Let two employees v_1 and v_2 control the same group $s_{H_1}(v_1) = s_{H_1}(v_2)$ in the hierarchy H_1. Acyclicity of the hierarchy implies that the employee v_1 does not control the employee v_2 or vice versa. Suppose v_1 does not control v_2. Then consider the immediate superior m_1 of the employee v_2. If v_1 is immediately subordinated to m_1 then the edge (v_2, m_1) can be removed with no hierarchy cost increase (see Lemma 4). If v_1 is not immediately subordinated to m_1 then the edge (v_2,m_1) can be replaced to the edge (v_1,m_1). The equality $s_{H_1}(v_1) = s_{H_1}(v_2)$ implies that the cost of the manager m_1 does not change. So, the cost of total hierarchy also does not change. Thus, in both cases the edge (v_2,m_1) can be removed. Similarly, we can remove all edges outcoming from v_2. After that the employee v_2 has no superiors and the employee v_2 can be removed with no hierarchy cost increase[73]. If in the obtained hierarchy some employees control the same group then we can repeat the removal described above. Finally we obtain the hierarchy H' with employees controlling differing groups. Thus, condition (i) holds for H'. The cost of H' is less than or equal to the cost of H_1: $c(H') \le c(H_1)$.

If some manager m_2 in the hierarchy H' has no superiors and controls the group $s_{H'}(m_2) \ne N$ then this manager can be removed with no hierarchy cost increase. We can repeat such removal. As a result, we obtain the hierarchy H''. In H'' any manager without superiors controls the group N. Definition 1 and condition (i)[74] imply that there is the single such manager m in the hierarchy H''.[75] At least one edge outcomes from any node $v \ne m$ in the hierarchy H''. Acyclicity implies that we can construct the path from v to m. So, all employees are subordinated to m. Thus, conditions (i) and (ii) hold for the hierarchy H''. The cost of the hierarchy H'' is less than or equal to the cost of H'. So, the cost of H'' is less than or equal to the cost of H_1: $c(H'') \le c(H_1)$.

Let the employees v_3 and v_4 be immediately subordinated to the common manager m_3 in the hierarchy H'' and the employee v_3 be subordinated to the employee v_4. Then $s_{H''}(v_3) \subseteq s_{H''}(v_4)$ (see Lemma 1). Lemma 4 implies that the edge (v_3, m_3) can be removed with no hierarchy cost increase. After removal the employee v_3 has at least one immediate superior because v_3 is subordinated to v_4. We can repeat such

[73] Definition 1 is fulfilled because the maximal group N is controlled by some manager (if v_2 controls the group N in the hierarchy H_1 then v_1 controls this group too).

[74] We have removed some managers without violation of condition (i).

[75] We cannot remove this manager because in this case Definition 1 is violated and the graph is not hierarchy controlling the set of workers N.

removal. As a result, we obtain the hierarchy H_2 in which condition (iii) holds. The modifications described above do not change groups controlled by the managers. The manager m is the only manager without superiors. Thus, conditions (i), (ii) and (iii) hold for the hierarchy H_2. Moreover, the cost of the hierarchy H_2 is less than or equal to the cost of H''. So, the cost of H_2 is less than or equal to the cost of H_1: $c(H_2) \le c(H_1)$.

Described above reconstructions do not increase the number of immediate subordinates of any manager. If H_1 is a r-hierarchy then H_2 is r-hierarchy too and conditions (i), (ii) and (iii) hold.

By definition conditions (ii) and (iii) hold for any tree.[76] Let H_1 be a tree.

Suppose there exists a manager m_4 with the single immediate subordinate v. So, the equality $s_{H_1}(v) = s_{H_1}(m_4)$ holds. Consider two cases.

1. If m_4 has the immediate superior m_5 then from condition (iii) we find that v is not subordinated immediately to m_5. Therefore, we can resubordinate the employee v immediately to the manager m_5 and remove the manager m_4 with no change the groups controlled by other managers.

2. If the manager m_4 has no superiors then he or she can be removed also. After this removal only the employee v has no superiors. Any other employee has exactly one immediate superior.

In both cases the obtained hierarchy is a tree (see Definition 2). And the cost of this tree is less than or equal to the cost of H_1. We can repeat such removal. As a result, we obtain the tree H_2 with each manager having at least two immediate subordinates. Moreover, the cost of the tree H_2 is less than or equal to the cost of H_1: $c(H_2) \le c(H_1)$. So, we have to prove only condition (i) for H_2.

Consider the manager m, which has no superiors in the tree H_2. The manager m controls the group N and has $k \ge 2$ immediately subor-

[76] Any tree has single manager without superiors. Acyclicity implies that there exists a path from any other employee to this manager. So, all other employees are subordinated to this manager. If one immediate subordinate v' of some manager controls other immediate subordinate v'' then v'' has two or more immediate superiors. It contradicts Definition 2.

dinated employees $v_1,\ldots,v_k$. Lemma 2 implies that these employees control non-overlapping groups. Therefore, any subordinate of v_i and any subordinate of v_j cannot control each other for any $i \neq j$. And any subordinate of v_i and any subordinate of v_j cannot control the same group. So, there are k independent subtrees with the roots $v_1,\ldots,v_k$ and we have to prove that condition (i) holds in these subtrees. Each of the employees $v_1,\ldots,v_k$ controls less group than the group N controlled by the manager m. Thus, we can use the method of induction by the size of the group, controlled by the root of the tree (by analogy with the proof of Lemma 2).

Therefore, conditions (i), (ii) and (iii) hold for the tree H_2.

The reconstructions described above do not increase the number of immediate subordinates of any manager. If H_1 is r-tree then H_2 is r-tree too and conditions (i), (ii) and (iii) hold.■

Proof of Proposition 2. Consider hierarchy $H=(M \cup N, E) \in \Omega(N)$. Let $M=\{m_1,\ldots,m_q\}$ be the set of managers of this hierarchy. Let $x_i = F_H^{\text{int}}(m_i) + F_H^{ext}(m_i)$ be the sum of internal and external flows of the manager m_i, $1 \le i \le q$. Let x be the sum of all flows inside the technological network and flows between the network and environment $x = \sum_{\{w',w''\} \subseteq N} f(w',w'') + \sum_{w \in N} f(w, w_{env})$. Any flow inside the network is controlled by one or more managers in the hierarchy. Top manager participates in control of all flows between the network and environment. So, the inequality $x_1 + \ldots + x_q \ge x$ holds.

There is single manager m in two-tier hierarchy. The sum of internal and external flows of the manager m equals x. So, the cost of two-tier hierarchy equals $\varphi(x)$. Cost of the hierarchy H equals $\varphi(x_1) + \ldots + \varphi(x_q)$. The function $\varphi(\cdot)$ subadditivity implies that the following inequality holds:

$$\varphi(x_1)+\varphi(x_2)+\ldots+\varphi(x_q) \ge \varphi(x_1+x_2)+\varphi(x_3)+\ldots+\varphi(x_q) \ge \ldots \ge \varphi(x_1+\ldots+x_q).$$

The inequality $x_1 + \ldots + x_q \ge x$ and non-decrease of the function $\varphi(\cdot)$ imply that the following inequality holds:

$$\varphi(x_1)+\varphi(x_2)+\ldots+\varphi(x_q) \ge \varphi(x).$$

So, the cost of two-tier hierarchy is less than or equal to cost of any other hierarchy. Thus, two-tier hierarchy is optimal.■

Proof of Lemma 5. By definition of concave function for each $z_1, z_2 \in R_+$ and each $\gamma \in [0;1]$ the inequality $\varphi(\gamma z_1 + (1-\gamma)z_2) \geq \gamma\varphi(z_1) + (1-\gamma)\varphi(z_2)$ holds. Let's prove that the inequality $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ holds for each $x, y \in R_+$. It is obvious for x=y=0. Let's define the values z_1=0, $z_2=x+y>0$. Consider the following values of γ: $y/(x+y)$ and $x/(x+y)$. So, the following inequalities hold:

$$\varphi(x) \geq \varphi(0)y/(x+y) + \varphi(x+y)x/(x+y);$$
$$\varphi(y) \geq \varphi(0)x/(x+y) + \varphi(x+y)y/(x+y).$$

Let's add these inequalities: $\varphi(x) + \varphi(y) \geq \varphi(0) + \varphi(x+y) \geq \varphi(x+y)$. Thus, for one-dimensional flows concave cost function $\varphi(\cdot)$ is subadditive. ■

Proof of Proposition 3. Consider a set of managers $M=\{m_1,\ldots,m_q\}$ controlling all flows inside symmetric process line with minimal total costs. Managers from set M can immediately control the workers or can be organized in more complex multi-tier structure. But we do not suppose that managers in M are organized in hierarchy. Let k_i be the number of internal flows controlled by the manager m_i, $1 \leq i \leq q$. Let l_i be the number of external flows of the manager m_i (m_i participates in these flows control). Then the internal flow of the manager m_i equals $F^{\text{int}}(m_i) = \lambda k_i$ and external flow equals $F^{ext}(m_i) = \lambda l_i$. So, the cost of the manager m_i equals $\varphi((k_i + l_i)\lambda)$. Total costs of all managers in M equal $\varphi((k_1 + l_1)\lambda) + \ldots + \varphi((k_q + l_q)\lambda)$.

Let's prove that any manager m_i participates in control of two or more external flows. Let $w_k \in N$ be the worker with minimum number subordinated to the manager m_i. Then the flow $f(w_{k-1},w_k)$ (or $f(w_{env},w_1)$ for k=1) is external for the manager m_i. Similarly we can consider manager's m_i subordinated worker with maximum number. So, the inequality $l_i \geq 2$ holds.

Consider $n-1$ flows $f(w_{i-1}, w_i) = \lambda$ for each $2 \le i \le n$. Each flow controlled by one or more managers $m_1, \ldots, m_q$. Thus, the flow $f(w_{i-1}, w_i) = \lambda$ is internal for one or more managers. So, the inequality $k_1 + \ldots + k_q \ge n-1$ holds. Moreover $k_i \le n-1$.

Let's construct the tree $H = (N \cup M', E')$ in the following way. At the beginning there are no managers. In any tree each worker must be immediately subordinated to exactly one manager. Let's hire the manager m_1' and immediately subordinate to m_1' k_1+1 workers with minimal numbers: 1,2,…,k_1+1. So, $s_H(m_1') = \{w_1, \ldots, w_{k_1+1}\}$. The manager m_1' and each of the workers $w_{k_1+2}, \ldots, w_n$ must be immediately subordinated to exactly one manager. Thus, after hiring of manager m_1' we have $n-k_1$ non-subordinated employees. Let's hire the manager m_2' and immediately subordinate m_1' and k_2 workers $w_{k_1+2}, \ldots, w_{k_1+k_2+1}$ to m_2'. So, after hiring of manager m_2' we have $n-k_1-k_2$ non-subordinated employees. Let's repeat similar hire and subordination. In the end we can obtain the following two results:

1. In case of $k_1 + \ldots + k_q = n-1$ $q' = q$ managers are hired. The manager $m_{q'-1}'$ and $k_{q'}$ non-subordinated workers $w_{k_1+\ldots+k_{q'-1}+2}, \ldots, w_n$ are immediately subordinated to the manager $m_{q'}'$.

2. In case of $k_1 + \ldots + k_q > n-1$ $q' \le q$ managers are hired. The manager $m_{q'-1}'$ and no more than $k_{q'}$ non-subordinated workers $w_{k_1+\ldots+k_{q'-1}+2}, \ldots, w_n$ are immediately subordinated to the manager $m_{q'}'$.

In both cases the manager $m_{q'}'$ controls all workers. So, the tree $H \in \Omega(N)$ is constructed. By construction each manager in H controls the group of consecutive workers in the process line. Consider the manager m_i' for some $1 \le i \le q'$. Let $v_1, \ldots, v_j$ be all immediate subordinates of m_i'. Lemma 2 implies that the employees $v_1, \ldots, v_j$ control non-overlapping groups. Lemma 1 implies that the equality $s_H(m_i') = s_H(v_1) \cup \ldots \cup s_H(v_j)$ holds. So, the part $s_H(m_i')$ of the process line controlled by m_i' is divided into parts $s_H(v_1), \ldots, s_H(v_j)$ controlled by $v_1,\ldots,v_j$. Thus, the manager m_i' controls $j-1$ internal flows and participates in control of two external flows. The cost of the manager m_i' equals $\varphi((j+1)\lambda)$. The manager m_i' controls no more than k_i+1 immediate subordinates. Therefore, the cost of the manager m_i' is less than or equal to $\varphi((k_i+2)\lambda)$. So, the following inequality holds:

$$c(H)\le\varphi((k_1+2)\lambda)+\ldots+\varphi((k_q+2)\lambda)\le\varphi((k_1+l_1)\lambda)+\ldots+\varphi((k_q+l_q)\lambda).$$

There are no more than q managers in the tree H. Non-negativity of $\varphi(\cdot)$ implies that additional items do not decrease cost. $\varphi((k_i+2)\lambda)$ is the upper bound of the manager m_i' cost. So, the first inequality holds. Non-decrease of $\varphi(\cdot)$ and $l_i\ge 2$ proof the second inequality.

So, the constructed tree H has no more managers than the set M and the cost of a manager in H is less than or equal to the cost of corresponding manager in M. Thus, the cost of H is less than or equal to total costs of any managers which control all flows inside symmetric process line.

Let $\varphi(\cdot)$ be a convex function. Let k_i' be the number of immediate subordinates of the manager m_i', $1\le i\le q'$. If there exist two managers with the difference of their immediate subordinates numbers greater than 1 then for some $1\le i,j\le q'-1$ the inequality $k_i'+1<k_j'$ holds. In the described above tree construction we can hire managers $m_1',\ldots,m_{q'}'$ in any order with no cost change. At the beginning of the tree construction we can hire the manager m_i' with k_i' immediate subordinates. After that we can hire the manager m_j' with k_j' immediate subordinates. All other managers can be hired in any order. So, the numbers $k_1',\ldots,k_{q'}'$ are permuted in the new tree. Thus, in the new tree (after the permutation) the inequality $k_1'+1<k_2'$ holds. By construction of the tree the manager m_2' has immediately subordinated worker $w_{k_1'+2}$ near by the group $s_H(m_1')$. Then we can resubordinate $w_{k_1'+2}$ immediately to the manager m_1' instead of the manager m_2'. After that each manager still controls the group of consecutive workers in the process line. After the resubordination only managers' m_1' and m_2' costs have been

changed. The manager m'_1 has $k'_1 + 1$ immediate subordinates. The manager m'_2 has $k'_2 - 1$ immediate subordinates. Follow we prove that the resubordination cannot cause increasing of the tree cost. So, we can repeat similar resubordinations. In obtained tree the numbers of immediate subordinates of the managers m'_1 and m'_2 are equal or differ by one. The described above modifications reduce dispersion of the numbers $k'_1, ..., k'_{q'}$. Thus, we can repeat modifications. In obtained tree the numbers of immediate subordinates of all managers are equal or differ by one. The cost of obtained tree is less than or equal to the cost of initial tree.

To prove the proposition we only have to prove that the inequality $\varphi((k'_1 + 2)\lambda) + \varphi(k'_2\lambda) \le \varphi((k'_1 + 1)\lambda) + \varphi((k'_2 + 1)\lambda)$ holds. Let's define the following values $z_1 = k'_1 + 1$, $z_2 = k'_2 + 1$. By definition of convex function for each $z_1, z_2 \in R_+$ and each $\gamma \in [0;1]$ the inequality $\varphi((\gamma z_1 + (1-\gamma) z_2)\lambda) \le \gamma\varphi(z_1\lambda) + (1-\gamma)\varphi(z_2\lambda)$ holds.

Let's define the following values $\gamma_1 = (z_2 - z_1 - 1)/(z_2 - z_1)$, $\gamma_2 = 1/(z_2 - z_1)$. Then $0 < \gamma_1, \gamma_2 < 1$ and the following equalities hold: $\gamma_1 + \gamma_2 = 1$, $\gamma_1 z_1 + (1-\gamma_1) z_2 = z_1 + 1$, $\gamma_2 z_1 + (1-\gamma_2) z_2 = z_2 - 1$. Let's substitute γ_1, γ_2 in the above inequality:

$$\varphi((z_1 + 1)\lambda) \le \gamma_1\varphi(z_1\lambda) + (1-\gamma_1)\varphi(z_2\lambda),$$
$$\varphi((z_1 - 1)\lambda) \le \gamma_2\varphi(z_1\lambda) + (1-\gamma_2)\varphi(z_2\lambda).$$

Let's add these inequalities:

$$\varphi((z_1 + 1)\lambda) + \varphi((z_2 - 1)\lambda) \le \varphi(z_1\lambda) + \varphi(z_2\lambda).$$

So, the inequality $\varphi((k'_1 + 2)\lambda) + \varphi(k'_2\lambda) \le \varphi((k'_1 + 1)\lambda) + \varphi((k'_2 + 1)\lambda)$ holds. ■

Proof of Proposition 4. In expression (12) the cost of the tree depends on the value $\xi(r) = (r+1)^\alpha / (r-1)$. So, we have to minimize $\xi(r)$ to minimize the tree cost. Let's calculate the derivative of the function $\xi(r)$:

$$\xi'(r) = [\alpha(r+1)^{\alpha-1}(r-1) - (r+1)^\alpha]/(r-1)^2 =$$
$$= (r+1)^{\alpha-1}[(\alpha-1)r - \alpha - 1]/(r-1)^2.$$

The condition $\alpha > 1$ implies that the inequality $\xi'(r) < 0$ holds with $r < (\alpha+1)/(\alpha-1)$ and the inequality $\xi'(r) > 0$ holds with $r > (\alpha+1)/(\alpha-1)$. So, $r_0 = (\alpha+1)/(\alpha-1)$ is a single minimum point of the function $\xi(r)$. r is an integer number. So, the function $\xi(r)$ is minimized by one of the following integer numbers: either $r_- = \lfloor r_0 \rfloor$ (maximal integer is less than or equal to r_0) or $r_+ = \lceil r_0 \rceil$ (minimal integer is greater than or equal to r_0). If $\xi(r_-) < \xi(r_+)$ then $r_* = r_-$ is the minimum point of the function $\xi(r)$. If $\xi(r_-) \geq \xi(r_+)$ then $r_* = r_+$ is the minimum point of the function $\xi(r)$.

Thus, the function $\xi(r)$ is minimized by the value r_*. r_* is one of the two integer numbers nearest with $(\alpha+1)/(\alpha-1)$. If $(\alpha+1)/(\alpha-1)$ is an integer number then $r_* = r_- = r_+$.

The inequality $\xi(r) \geq \xi(r_*)$ *holds for any integer number* $r \geq 1$ because r_* is the minimum point of the function $\xi(r)$.

Let n–1 contain $r_* - 1$. Let H be any tree with each manager having exactly r_* immediate subordinates and controlling a group of consecutive workers in the process line. Expression (12) implies that the number of managers in H equals $(n-1)/(r_*-1)$ and the cost of H:

$$(r_* + 1)^\alpha \lambda^\alpha (n-1)/(r_* - 1) = (n-1)\lambda^\alpha \xi(r_*) . \qquad (*)$$

Below we prove that the cost of an optimal hierarchy is greater than or equal to (*) for any n. So, H is an optimal hierarchy and the cost equals (*) if n–1 contains $r_* - 1$. Moreover Proposition 3 implies that the cost of an optimal hierarchy is less than or equal to costs of any managers which control all the flows inside symmetric process line. Therefore, for any n expression (12) with $r = r_*$ is a lower bound of controlling cost.

To prove the proposition we have to prove that the cost of an optimal hierarchy is greater than or equal to (*) for any n. The power cost function is convex because of $\alpha > 1$. Convexity and Proposition 3 imply that there exists an optimal tree H^*. Numbers of immediate subordinates of all managers in H^* are equal or differ by one. Moreover, in H^* each manager controls the group of consecutive workers in the process line. Let $m_1, \ldots, m_q$ be all managers of H^*. Then $q_1 > 0$ managers have r immediate subordinates and q_2 managers have $r+1$

immediate subordinates, where r, q_1, q_2 are integer numbers such that $2 \le r \le n$, q_1+q_2=q, $2 \le r \le n$. Conditions (9) implies that the following equality hold:

$$(r-1)q + q_2 = n-1. \qquad (**)$$

Thus, $q = (n-1)/(r-1) - q_2/(r-1)$. Expression (11) implies that costs of q_1 managers equal $q_1(r+1)^\alpha \lambda^\alpha$, costs of q_2 managers equal $q_2(r+2)^\alpha \lambda^\alpha$. Therefore, the following equalities hold:

$$c(H^*) = [q_1(r+1)^\alpha + q_2(r+2)^\alpha]\lambda^\alpha = [(q-q_2)(r+1)^\alpha + q_2(r+2)^\alpha]\lambda^\alpha =$$

$$= [(\frac{n-1}{r-1} - \frac{rq_2}{r-1})(r+1)^\alpha + q_2(r+2)^\alpha]\lambda^\alpha =$$

$$= [(n-1)\xi(r) + q_2((r+2)^\alpha - r(r+1)^\alpha/(r-1))]\lambda^\alpha.$$

If $(r+2)^\alpha - r(r+1)^\alpha/(r-1) \ge 0$ then $\xi(r) \ge \xi(r_*)$ proofs the inequality $c(H^*) \ge (n-1)\lambda^\alpha \xi(r_*)$. So, the cost of optimal hierarchy is greater than or equal to (*) for any n.

We have to consider the case $(r+2)^\alpha - r(r+1)^\alpha/(r-1) < 0$. Let's obtain lower bound for $c(H^*)$ using the following upper bound for q_2. Equality (**) implies that the equality $q_2 = n-1-(r-1)q$ holds. The maximum of $n-1-(r-1)q$ corresponds with the minimum q. Taking into account $q \ge q_2$ we can write $q_2 \le n-1-(r-1)q_2$. Thus, $q_2 \le (n-1)/r$. Let's substitute this upper bound in $c(H^*)$:

$$c(H^*) \ge [(n-1)\xi(r) + \frac{n-1}{r}((r+2)^\alpha - r(r+1)^\alpha/(r-1))]\lambda^\alpha =$$

$$= [(n-1)\xi(r) - (n-1)\xi(r) + (n-1)(r+2)^\alpha / r]\lambda^\alpha = (n-1)\xi(r+1)\lambda^\alpha.$$

The inequality $\xi(r+1) \ge \xi(r_*)$ holds because r_* is the minimum point of the function $\xi(\cdot)$ for all integers $r \ge 1$. Thus, $c(H^*) \ge (n-1)\lambda^\alpha \xi(r_*)$. So, the cost of an optimal hierarchy is greater than or equal to (*) for any n. ■

Proof of Proposition 5. Consider an optimal hierarchy $H = (N \cup M, E) \in \Omega(N)$ controlling process lines with functional links. In compliance with Section 2.6 (see cost function (26) on page 63) in H any product flow is controlled by divisional manager or strategic manager controlling interactions between departments. Any functional flow is controlled by functional manager or strategic manager controlling interactions between divisions.

Suppose at least in one process line N_i the flow $f(w_{i,j},w_{i,j+1})$ is not controlled by divisional manager. Then this flow is controlled by the strategic manager controlling interactions between departments j and $j+1$. So, the departments j and $j+1$ are organized in H and some strategic manager controls all product flows between these departments. Let $j_1,j_2,\ldots, j_{n_1}$ be all such indexes j. n_1 is a number of indexes j with flow $f(w_{i,j},w_{i,j+1})$ not controlled by any divisional manager at least in one process line N_i, $0 \le n_1 \le n-1$. If $n_1=0$ then all flows are controlled by divisional managers. If $n_1=n-1$ then for each $1 \le j \le n-1$ the flow $f(w_{i,j},w_{i,j+1})$ is not controlled by divisional manager at least in one process line N_i. Let's estimate costs of functional, strategic and divisional managers in the hierarchy H.

1. Suppose $n_1>0$. If indexes $j_1,j_2,\ldots, j_{n_1}$ are consecutive numbers then there are n_1+1 or more departments in H. If not then there are more than n_1+1 departments in H (up to $2n_1$). Thus, the hierarchy H contains at least n_1+1 departments. Each department controls the functional line with l workers and flow intensity θ. Expression (21) implies that *costs of functional managers are greater than or equal to the following value*:

$$x_1 = (n_1+1)(l-1)(\theta^\alpha + c_0^\alpha)(r_*+1)^\alpha/(r_*-1).$$

If $n_1=0$ then the lower bound x_1 is not used below.

2. If $n_1>0$ then item 1 leads to the fact that there are at least n_1+1 departments in the hierarchy H. If indexes $j_1,j_2,\ldots, j_{n_1}$ are consecutive numbers then department chiefs are linked in line with product flow intensity $l\lambda$. These flows (department interactions) are controlled by strategic managers. Expression (25) implies that the minimal costs of strategic managers controlling interactions between departments are equal to the following value:

$$x_2 = n_1((l\lambda)^\alpha + c_0^\alpha)(r_*+1)^\alpha/(r_*-1).$$

If indexes $j_1,j_2,\ldots, j_{n_1}$ are inconsecutive numbers then all set of indexes can be divided into sets of consecutive indexes. Let $k_1,\ldots,k_t$ be the numbers of indexes in each set, $k_1+\ldots+k_t=n_1$. The first set corresponds with k_1+1 department. Therefore, in the expression x_2 the number n_1 is replaced by k_1. For the second set the number n_1 is replaced by k_2, etc. Total costs for all sets equal to x_2. Thus, x_2 is *the minimal costs of strategic managers controlling interactions between departments*. If n_1=0 then x_2=0 is a lower bound too.

3. If $n_1<n-1$ then in each process line there are at least $n-1-n_1$ product flows controlled by divisional managers. If these flows are consecutive then the divisional managers control the part of process line with at least $n-1-n_1$ product flows inside the part. So, the part contains at least $n-n_1$ workers. Each of l process lines contains such part. The flow intensity equals λ. Expression (19) implies that costs of divisional managers are greater than or equal to the following value:

$$x_3 = l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha)(r_* + 1)^\alpha / (r_* - 1).$$

If described above flows are not consecutive then these flows can be divided into sets of consecutive flows. Let $k_1,\ldots,k_t$ be the numbers of flows in each set, $k_1+\ldots+k_t= n-1-n_1$. The first set corresponds with the part of process line with k_1+1 workers. Therefore, in the expression x_3 the number $n-n_1-1$ is replaced by k_1. For the second set the number $n-n_1-1$ is replaced by k_2, etc. Total costs for all sets equal to x_3. Thus, x_3 is *the minimal costs of divisional managers*. If $n_1=n-1$ then x_3=0 is a lower bound too.

Similar reasoning is true for functional flows. Let's repeat it briefly. Suppose at least in one functional line N^j the flow $f(w_{i,j},w_{i+1,j})$ is not controlled by the functional manager. Then this flow is controlled by the strategic manager controlling interactions between divisions i and i+1. So, the divisions i and i+1 are organized in H and some strategic manager controls all functional flows between these divisions. Let l_1 be the number of indexes i with flow $f(w_{i,j},w_{i+1,j})$ not controlled by any functional manager at least in one functional line N^j, $0 \le l_1 \le l-1$. Let's estimate costs of divisional, strategic and functional managers in the hierarchy H.

1. Suppose $l_1>0$. The hierarchy H contains at least l_1+1 divisions. Each division controls the process line with n workers and flow intensity λ. Expression (19) implies that *costs of divisional managers are greater than or equal to the following value*:

$$y_1 = (l_1 + 1)(n-1)(\lambda^\alpha + c_0^\alpha)(r_* + 1)^\alpha / (r_* - 1)\,.$$

If l_1=0 then the lower bound y_1 is not used below.

2. If l_1>0 then item 1 leads to the fact that there are at least l_1+1 divisions in the hierarchy *H*. Division chiefs are linked in line with functional flow intensity $n\theta$. Expression (23) implies that *costs of strategic managers controlling interactions between divisions are greater than or equal to the following value*:

$$y_2 = l_1((n\theta)^\alpha + c_0^\alpha)(r_* + 1)^\alpha / (r_* - 1)\,.$$

If l_1=0 then y_2=0 is a lower bound too.

3. If l_1<l–1 then in each functional line there are at least l–1–l_1 functional flows controlled by functional managers. If these flows are consecutive then the functional managers control the part of functional line with at least l–1–l_1 functional flows inside the part. So, the part contains at least l–l_1 workers. Each of n functional lines contains such part. The flow intensity equals θ. Expression (21) implies that *costs of functional managers are greater than or equal to the following value*:

$$y_3 = n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha)(r_* + 1)^\alpha / (r_* - 1)\,.$$

If described above flows are not consecutive then we can repeat the reasoning described above for the value x_3. If l_1=l–1 then y_3=0 is a lower bound too.

Therefore, in the hierarchy *H* total strategic managers costs is greater than or equal to x_2+y_2. And we have two lower bounds x_1 and y_3 for costs of functional managers and two lower bounds y_1 and x_3 for costs of divisional managers. Thus, the following inequalities hold:

$$c(H) \geq x_2 + x_3 + y_2 + y_3 \text{ and } c(H) \geq x_1 + x_2 + y_1 + y_2\,. \qquad (*)$$

The lower bound $c(H) \geq x_1 + x_2 + y_1 + y_2$ can be used only in case n_1>0 and l_1>0. To prove the proposition it is enough to prove that one of lower bounds (*) is greater than or equal to the cost of divisional, functional or matrix hierarchy. Expressions (27), (28), (29) imply that the following equalities hold:

$$c(H_{divisional}) = [l(n-1)(\lambda^\alpha + c_0^\alpha) + (l-1)((n\theta)^\alpha + c_0^\alpha)](r_* + 1)^\alpha / (r_* - 1)\,,$$

$$c(H_{functional}) = [n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha)](r_* + 1)^\alpha / (r_* - 1)\,,$$

$$c(H_{matrix}) = [l(n-1)(\lambda^\alpha + c_0^\alpha) + n(l-1)(\theta^\alpha + c_0^\alpha)](r_* + 1)^\alpha / (r_* - 1)\,.$$

Below the multiplier $(r_* + 1)^\alpha / (r_* - 1)$ is omitted because this multiplier is in all expressions and is of no importance. We consider sequentially the cases with matrix, divisional, and functional hierarchy cost minimizing. The lower bound $c(H) \geq x_2 + x_3 + y_2 + y_3$ from (*) will be used.

1. Suppose the inequalities $c(H_{matrix}) \leq c(H_{divisional})$ and $c(H_{matrix}) \leq c(H_{functional})$ hold. Thus, the following inequalities $((n\theta)^\alpha + c_0^\alpha) \geq n(\theta^\alpha + c_0^\alpha)$ and $((l\lambda)^\alpha + c_0^\alpha) \geq l(\lambda^\alpha + c_0^\alpha)$ hold too. The inequality $x_2 + x_3 + y_2 + y_3 \geq c(H_{matrix})$ is given by:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) +$$
$$+ n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \geq l(n-1)(\lambda^\alpha + c_0^\alpha) + n(l-1)(\theta^\alpha + c_0^\alpha).$$

To prove the inequality it is enough to substitute in the first member the expressions $((n\theta)^\alpha + c_0^\alpha) \geq n(\theta^\alpha + c_0^\alpha)$ and $((l\lambda)^\alpha + c_0^\alpha) \geq l(\lambda^\alpha + c_0^\alpha)$.

2. Suppose the inequalities $c(H_{matrix}) \leq c(H_{divisional})$ and $c(H_{matrix}) \geq c(H_{functional})$ hold. Thus, the following inequalities $((n\theta)^\alpha + c_0^\alpha) \geq n(\theta^\alpha + c_0^\alpha)$ and $l(\lambda^\alpha + c_0^\alpha) \geq ((l\lambda)^\alpha + c_0^\alpha)$ hold too. The inequality $x_2 + x_3 + y_2 + y_3 \geq c(H_{functional})$ is given by:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) +$$
$$+ n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \geq n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha).$$

To prove the inequality it is enough to substitute in the first member the expressions $((n\theta)^\alpha + c_0^\alpha) \geq n(\theta^\alpha + c_0^\alpha)$ and $l(\lambda^\alpha + c_0^\alpha) \geq ((l\lambda)^\alpha + c_0^\alpha)$.

3. Suppose the inequalities $c(H_{matrix}) \geq c(H_{divisional})$ and $c(H_{matrix}) \leq c(H_{functional})$ hold. Thus, the following inequalities $n(\theta^\alpha + c_0^\alpha) \geq ((n\theta)^\alpha + c_0^\alpha)$ and $((l\lambda)^\alpha + c_0^\alpha) \geq l(\lambda^\alpha + c_0^\alpha)$ hold too. The inequality $x_2 + x_3 + y_2 + y_3 \geq c(H_{divisional})$ is given by:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) +$$
$$+ n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \geq l(n-1)(\lambda^\alpha + c_0^\alpha) + (l-1)((n\theta)^\alpha + c_0^\alpha).$$

To prove the inequality it is enough to substitute in the first member the expressions $n(\theta^\alpha + c_0^\alpha) \geq ((n\theta)^\alpha + c_0^\alpha)$ and $((l\lambda)^\alpha + c_0^\alpha) \geq l(\lambda^\alpha + c_0^\alpha)$.

We have to consider only the case $c(H_{matrix}) > c(H_{divisional})$ and $c(H_{matrix}) > c(H_{functional})$. So, the following inequalities hold below:

$$n(\theta^\alpha + c_0^\alpha) > ((n\theta)^\alpha + c_0^\alpha),\ l(\lambda^\alpha + c_0^\alpha) > ((l\lambda)^\alpha + c_0^\alpha) \qquad (**).$$

1. Consider the case $c(H_{divisional}) \le c(H_{functional})$.

a) The lower bound $c(H) \ge x_2 + x_3 + y_2 + y_3$ from (*) will be used. The inequality $x_2 + x_3 + y_2 + y_3 \ge c(H_{divisional})$ is given by:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) +$$
$$+ n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \ge l(n-1)(\lambda^\alpha + c_0^\alpha) + (l-1)((n\theta)^\alpha + c_0^\alpha).$$

Let's group the first item on the right with the second item on the left, and the second item on the right with the third item on the left:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \ge l(\lambda^\alpha + c_0^\alpha)n_1 + (l - l_1 - 1)((n\theta)^\alpha + c_0^\alpha).$$

Let's group the first item on the right with the first item on the left, and the second item on the left with the second item on the right:

$$(l - l_1 - 1)[n(\theta^\alpha + c_0^\alpha) - ((n\theta)^\alpha + c_0^\alpha)] \ge n_1[l(\lambda^\alpha + c_0^\alpha) - ((l\lambda)^\alpha + c_0^\alpha)].$$

The inequality $c(H_{divisional}) \le c(H_{functional})$ leads to:

$$l(n-1)(\lambda^\alpha + c_0^\alpha) + (l-1)((n\theta)^\alpha + c_0^\alpha) \le n(l-1)(\theta^\alpha + c_0^\alpha) +$$
$$+ (n-1)((l\lambda)^\alpha + c_0^\alpha).$$

So, we can evaluate the lower bound of the expression in square brackets in the left-hand member of the inequality. The lower bound is given by $(n-1)[l(\lambda^\alpha + c_0^\alpha) - ((l\lambda)^\alpha + c_0^\alpha)]/(l-1)$. Let's substitute the lower bound in the inequality:

$$(l - l_1 - 1)(n-1) \ge (l-1)n_1.$$

The expression in the square brackets is positive (see (**)). So, this expression is cancelled. Thus, if the inequality $(l - l_1 - 1)/(l-1) \ge n_1/(n-1)$ holds then $c(H) \ge c(H_{divisional})$.

b) Consider the case $(l - l_1 - 1)/(l-1) < n_1/(n-1)$. If n_1=0 or l_1=0 then this condition is violated because $l_1 \le l-1$ and $n_1 \le n-1$. Therefore, in the concerned case n_1>0 and l_1>0. So, the lower bound $c(H) \ge x_1 + x_2 + y_1 + y_2$ from (*) can be used.

The inequality $x_1 + x_2 + y_1 + y_2 \ge c(H_{divisional})$ is given by:

$$(n_1+1)(l-1)(\theta^\alpha+c_0^\alpha)+n_1((l\lambda)^\alpha+c_0^\alpha)+(l_1+1)(n-1)(\lambda^\alpha+c_0^\alpha)+$$
$$+l_1((n\theta)^\alpha+c_0^\alpha)\ge l(n-1)(\lambda^\alpha+c_0^\alpha)+(l-1)((n\theta)^\alpha+c_0^\alpha).$$

Let's group the first item on the right with the third item on the left, and the second item on the right with the fourth item on the left:

$$(n_1+1)(l-1)(\theta^\alpha+c_0^\alpha)+n_1((l\lambda)^\alpha+c_0^\alpha)\ge$$
$$\ge(l-l_1-1)[(n-1)(\lambda^\alpha+c_0^\alpha)+((n\theta)^\alpha+c_0^\alpha)].$$

The inequality $c(H_{divisional})\le c(H_{functional})$ leads to:

$$l(n-1)(\lambda^\alpha+c_0^\alpha)+(l-1)((n\theta)^\alpha+c_0^\alpha)\le n(l-1)(\theta^\alpha+c_0^\alpha)+$$
$$+(n-1)((l\lambda)^\alpha+c_0^\alpha).$$

We can substitute *l*–1 instead of *l* in the first item. So, we can evaluate the upper bound of the expression in square brackets in the right-hand member of the inequality. The upper bound is given by:

$$[(n-1)(\lambda^\alpha+c_0^\alpha)+((n\theta)^\alpha+c_0^\alpha)]\le$$
$$\le(n(l-1)(\theta^\alpha+c_0^\alpha)+(n-1)((l\lambda)^\alpha+c_0^\alpha))/(l-1).$$

Let's substitute the upper bound in the inequality:

$$(n_1+1)(l-1)(\theta^\alpha+c_0^\alpha)+n_1((l\lambda)^\alpha+c_0^\alpha)\ge$$
$$\ge(l-l_1-1)[n(l-1)(\theta^\alpha+c_0^\alpha)+(n-1)((l\lambda)^\alpha+c_0^\alpha)]/(l-1).$$

Let's group the items:

$$(\theta^\alpha+c_0^\alpha)[(n_1+1)(l-1)-n(l-l_1-1)]\ge$$
$$\ge((l\lambda)^\alpha+c_0^\alpha)[(l-l_1-1)(n-1)/(l-1)-n_1].$$

We consider the case $(l-l_1-1)/(l-1)<n_1/(n-1)$. Therefore, the right-hand member is negative. Moreover $n_1(l-1)>(l-l_1-1)(n-1)$. Let's add *l*–1 to both members:

$$(n_1+1)(l-1)>(l-l_1-1)n-l+l_1+1+l-1=(l-l_1-1)n+l_1.$$

So, the left-hand member of the inequality is non-negative. Thus, in the case $(l-l_1-1)/(l-1)<n_1/(n-1)$ the inequality $c(H)\ge c(H_{divisional})$ holds too.

2. Similarly consider the last case $c(H_{functional})\le c(H_{divisional})$.

a) It follows from (*) that the inequality $c(H)\ge x_2+x_3+y_2+y_3$ holds. The inequality $x_2+x_3+y_2+y_3\ge c(H_{functional})$ is given by:

$$n_1((l\lambda)^\alpha + c_0^\alpha) + l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) +$$
$$+ n(l - l_1 - 1)(\theta^\alpha + c_0^\alpha) \geq n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha).$$

Let's group the first item on the right with the fourth item on the left, and the second item on the right with the first item on the left:

$$l(n - n_1 - 1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) \geq n(\theta^\alpha + c_0^\alpha)l_1 + (n - n_1 - 1)((l\lambda)^\alpha + c_0^\alpha).$$

Let's group the first item on the right with the second item on the left, and the first item on the left with the second item on the right:

$$(n - n_1 - 1)[l(\lambda^\alpha + c_0^\alpha) - ((l\lambda)^\alpha + c_0^\alpha)] \geq l_1[n(\theta^\alpha + c_0^\alpha) - ((n\theta)^\alpha + c_0^\alpha)].$$

The inequality $c(H_{functional}) \leq c(H_{divisional})$ leads to:

$$n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha) \leq l(n-1)(\lambda^\alpha + c_0^\alpha) +$$
$$+ (l-1)((n\theta)^\alpha + c_0^\alpha).$$

So, we can evaluate the lower bound of the expression in square brackets in the left-hand member of the inequality. The lower bound is given by $(l-1)[n(\theta^\alpha + c_0^\alpha) - ((n\theta)^\alpha + c_0^\alpha)]/(n-1)$. Let's substitute the lower bound in the inequality:

$$(n - n_1 - 1)(l-1) \geq (n-1)l_1 .$$

The expression in the square brackets is positive (see (**)). So, this expression is cancelled. Thus, if the inequality $(n - n_1 - 1)/(n-1) \geq l_1/(l-1)$ holds then $c(H) \geq c(H_{functional})$.

b) Consider the case $(n - n_1 - 1)/(n-1) < l_1/(l-1)$. If l_1=0 or n_1=0 then this condition is violated because $n_1 \leq n-1$ and $l_1 \leq l-1$. Therefore, in the concerned case n_1>0 and l_1>0. So, the lower bound $c(H) \geq x_1 + x_2 + y_1 + y_2$ from (*) can be used. The inequality $x_1 + x_2 + y_1 + y_2 \geq c(H_{functional})$ is given by:

$$(n_1 + 1)(l-1)(\theta^\alpha + c_0^\alpha) + n_1((l\lambda)^\alpha + c_0^\alpha) + (l_1 + 1)(n-1)(\lambda^\alpha + c_0^\alpha) +$$
$$+ l_1((n\theta)^\alpha + c_0^\alpha) \geq n(l-1)(\theta^\alpha + c_0^\alpha) + (n-1)((l\lambda)^\alpha + c_0^\alpha).$$

Let's group the first item on the right with the first item on the left, and the second item on the left with the second item on the right:

$$(l_1 + 1)(n-1)(\lambda^\alpha + c_0^\alpha) + l_1((n\theta)^\alpha + c_0^\alpha) \geq$$
$$\geq (n - n_1 - 1)[(l-1)(\theta^\alpha + c_0^\alpha) + ((l\lambda)^\alpha + c_0^\alpha)].$$

The inequality $c(H_{functional}) \leq c(H_{divisional})$ is given by:

$$n(l-1)(\theta^{\alpha}+c_0^{\alpha})+(n-1)((l\lambda)^{\alpha}+c_0^{\alpha})\le l(n-1)(\lambda^{\alpha}+c_0^{\alpha})+$$
$$+(l-1)((n\theta)^{\alpha}+c_0^{\alpha}).$$

We can substitute n–1 instead of n in the first item. So, we can evaluate the upper bound of the expression in square brackets in the right-hand member of the inequality. The upper bound is given by:

$$[(l-1)(\theta^{\alpha}+c_0^{\alpha})+((l\lambda)^{\alpha}+c_0^{\alpha})]\le$$
$$\le (l(n-1)(\lambda^{\alpha}+c_0^{\alpha})+(l-1)((n\theta)^{\alpha}+c_0^{\alpha}))/(n-1).$$

Let’s substitute the upper bound in the inequality:

$$(l_1+1)(n-1)(\lambda^{\alpha}+c_0^{\alpha})+l_1((n\theta)^{\alpha}+c_0^{\alpha})\ge$$
$$\ge (n-n_1-1)[l(n-1)(\lambda^{\alpha}+c_0^{\alpha})+(l-1)((n\theta)^{\alpha}+c_0^{\alpha})]/(n-1).$$

Let’s group the items:

$$(\lambda^{\alpha}+c_0^{\alpha})[(l_1+1)(n-1)-l(n-n_1-1)]\ge$$
$$\ge ((n\theta)^{\alpha}+c_0^{\alpha})[(n-n_1-1)(l-1)/(n-1)-l_1].$$

We consider the case $(n-n_1-1)/(n-1)<l_1/(l-1)$. Therefore, the right-hand member is negative. Moreover $l_1(n-1)>(n-n_1-1)(l-1)$. Let’s add n–1 to both members:

$$(l_1+1)(n-1)>(n-n_1-1)l-n+n_1+1+n-1=(n-n_1-1)l+n_1.$$

So, the left-hand member of the inequality is non-negative. Thus, in the case $(n-n_1-1)/(n-1)<l_1/(l-1)$ the inequality $c(H)\ge c(H_{functional})$ holds too.

Thus, in all cases the cost of the hierarchy H is greater than or equal to the cost of divisional, functional or matrix hierarchy. Therefore, the inequality $c(H)\ge \min(c(H_{divisional}),c(H_{functional}),c(H_{matrix}))$ holds. So, divisional or functional or matrix hierarchy is optimal. ■

Proof of Lemma 6. Consider the function $\xi(\alpha)=[(n^{\alpha}-n)/(n-1)]^{1/\alpha}$. Conditions $\alpha>1$ and $n\ge 2$ imply that the inequality $\xi(\alpha)>0$ holds. Let’s find the logarithm and differentiate by α:

$$\alpha\ln\xi(\alpha)=\ln(n^{\alpha}-n)-\ln(n-1),$$
$$\alpha\xi'(\alpha)/\xi(\alpha)=(n^{\alpha}\ln n)/(n^{\alpha}-n).$$

It follows from $\alpha > 1$ that the inequality $\xi'(\alpha) > 0$ holds. Thus, $\xi(\alpha)$ is monotonously increasing by α.

To prove that the value $[(n^\alpha - n)/(n-1)]^{1/\alpha}$ is monotonously increasing by n it is enough to prove that the function $\varsigma(n) = (n^\alpha - n)/(n-1)$ is monotonously increasing by *n*. Let's differentiate by *n*:

$$\varsigma'(n) = [(\alpha n^{\alpha-1} - 1)(n-1) - (n^\alpha - n)]/(n-1)^2 .$$

We have to prove the inequality $\varsigma'(n) > 0$ or non-negativity of the expression in the square brackets:

$$(\alpha - 1)n^\alpha - (\alpha n^{\alpha-1} - 1) > 0 .$$

Left-hand member is monotonously increasing by *n* because it's derivative is positive: $\alpha(\alpha-1)n^{\alpha-2}(n-1) > 0$. If n=1 then the left-hand member equals to zero. If $n \geq 2$ the inequality holds. ■

Proof of Proposition 6. Proposition 1 implies that there exists an optimal hierarchy $H = (N \cup M, E) \in \Omega(N)$, which satisfies conditions (i)-(iii) (see page 24).

If each of the employees except the top manager has exactly one immediate superior then *H* is an optimal tree (see Definition 2 on page 16). Otherwise there exists an employee $v \in N \cup M$ with two or more immediate superiors. If there are several such employees then let's consider the employee on the highest tier. So, each of the superiors of the employee *v* except the top manager has exactly one immediate superior.

Let v_1 and u_1 be some different immediate superiors of the employee *v*. Condition (ii) of Proposition 1 implies that the employees v_1 and u_1 are subordinated to the top manager *m*. Thus, there exists the path from v_1 to *m* and the path from u_1 to *m*.[77] Therefore, there exists two different paths from *v* to *m*. These paths diverge in common node *v* and converge in other node *u* (in *m* or one of subordinates of the manager *m*). Let $v - v_1 - \ldots - v_{n_1}$ and $v - u_1 - \ldots - u_{n_2}$ be the parts of these paths from *v* to *u*. These parts have common first node *v*, common last node $v_{n_1} = u_{n_2} = u$ and different intermediate nodes. It follows from

[77] One of these paths can contain one node if v_1=m or u_1=m.

choice of the node v that each of the managers $v_1,\ldots,v_{n_1-1}$ has exactly one immediate superior – the next node in the path. This is true for the managers $u_1,\ldots,u_{n_2-1}$ too. Corresponding fragment of the hierarchy is shown in Figure 45.

Initial hierarchy H satisfies conditions (i)-(iii) of Proposition 1. Below we describe the reconstruction that does not increase the cost of the hierarchy. After each reconstruction obtained hierarchy will be denoted H just as the initial hierarchy. All reconstructed hierarchies satisfy condition (ii) of Proposition 1. So, all employees are subordinated to the top manager m. Therefore, different paths from v converge and the fragment of any reconstructed hierarchy looks like the fragment in Figure 45.

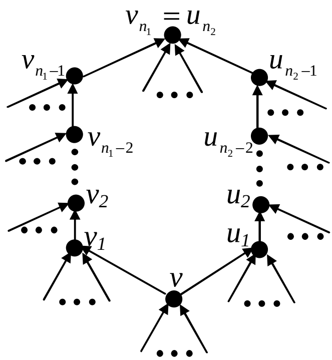

Figure 45. Optimal Hierarchy Reconstruction with Group-Monotonic Cost Function

There are two possible options of hierarchy H reconstruction (Figure 45 explains these options).

a) Suppose $s_H(v)=s_H(v_1)$.[78] So, the employees v and v_1 control the same group of workers. Let's remove the manager v_1. If v is not immediately subordinated to the manager v_2 then let's immediately subordinate the employee v to the manager v_2 instead of the manager v_1. After removal the groups controlled by the managers are not modified. So,

[78] In some cases reconstructed hierarchies do not satisfy condition (i) of Proposition 1. So, the equality $s_H(v)=s_H(v_1)$ can hold.

only the cost of the manager v_2 can be modified. This cost does not increase because of group-monotony. Thus, the obtained hierarchy is optimal.

After v_1 removal some employees may have no superiors. Such employee is not a worker because all workers are subordinated to the top manager. So, after v_1 removal in addition to the top manager some other managers may have no superiors. Such managers can be removed. The obtained graph is an optimal hierarchy. After removal new managers may have no superiors. These managers can be removed too, etc. Finiteness of M implies that we obtain the optimal hierarchy with only top manager having no superiors. Thus, condition (ii) of Proposition 1 holds.

b) Suppose $s_H(v) \neq s_H(v_1)$. So, the manager v_1 controls wider group then the employee v: $s_H(v) \subset s_H(v_1)$. Thus, v_1 has at least two immediate subordinates. Let's remove the edge (v,v_1). After removal the manager v_1 still has subordinates. The group $s_1=s_H(v_1)$ controlled by the manager v_1 can be changed to the new group s_1' if some workers from the group $s_H(v)$ are not controlled by the manager v_1 after removal. However, v_1 controls workers from the group s_1 which are not part of the group $s_H(v)$. Thus, $s_1' \subseteq s_1$, $(s_1 \setminus s_1') \subseteq s_H(v)$. There is exactly one edge outgoing from the node v_1. The modification of the group $s_1=s_H(v_1)$ can cause the modification of the group $s_2=s_H(v_2)$ controlled by the manager v_2. Let s_2' be the modified group. As described above only workers from $s_H(v)$ can be removed from the group s_1. So, only such workers can be removed from the group s_2. Thus, $s_2' \subseteq s_2$, $(s_2 \setminus s_2') \subseteq s_H(v)$. Similarly for each $i = \overline{3, n_1 - 1}$ the group $s_i=s_H(v_i)$ controlled by the manager v_i changes to the group s_i', $s_i' \subseteq s_i$, $(s_i \setminus s_i') \subseteq s_H(v)$.

Consider the group $s_H(v_{n_1})$. This group equals to the union of the groups controlled by all the immediate subordinates of the manager v_{n_1} (see Lemma 1 on page 16). Among these groups only the group s_{n_1-1} controlled by the manager v_{n_1-1} can be changed after the edge (v,v_1)

removal.[79] It follows from $(s_{n_1-1} \setminus s'_{n_1-1}) \subseteq s_H(v)$ that only workers from the group $s_H(v)$ can be removed from the group s_{n_1-1}. However, these workers are the part of the group $s_H(u_{n_2-1})$. Thus, the group $s_H(v_{n_1})$ is not changed. Therefore, the groups controlled by the superiors of the manager v_{n_1} are not changed too.

So, removal of the edge (v,v_1) can change the groups $s_H(v_1),...,s_H(v_{n_1-1})$ only. Thus, the top manager still controls all the workers, each manager has subordinates and the obtained graph is acyclic (edge removal cannot cause cycles). Therefore, the obtained graph satisfies all conditions of Definition 1. We obtain the hierarchy.

Moreover, each employee except the top manager has at least one immediate superior. All employees are subordinated to the top manager because of acyclicity. So, the hierarchy satisfies condition (ii) of Proposition 1.

The number of employees immediately subordinated to the manager v_1 decreases by one. The number of employees subordinated to each of the managers $v_2,\dots,v_{n_1}$ does not change. However, the group controlled by immediate subordinate of the manager v_i can be reduced, $i=\overline{2,n_1}$. So, costs of managers $v_2,\dots,v_{n_1}$ do not increase because of group-monotony. Thus, the cost of total hierarchy does not increase too. So, the obtained hierarchy is optimal.

Both in the option a) and in the option b) we obtain the optimal hierarchy satisfying condition (ii) of Proposition 1. Therefore, we can repeat the reconstruction a) or b) while there is an employee with two or more immediate superiors. After each reconstruction the number of edges decreases at least by one. Finiteness of the edge set E implies that the reconstructions come to an end after finite number of steps.

In the obtained optimal hierarchy H_1 only top manager has no superiors. Each of the other employees in H_1 has exactly one immediate superior. So, H_1 is an optimal tree. Proposition 1 implies that there

[79] Among manager's v_{n_1} immediate subordinates only v_{n_1-1} controls the managers $v_1,\dots v_{n_1-2}$ because each of them has only one immediate superior.

exists a tree H^* satisfying conditions (i)-(iii) (see page 24). Moreover, the cost of H^* is less than or equal to the cost of H_1.

Thus, H^* is an optimal tree satisfying conditions (i)-(iii). ■

Proof of Proposition 7. Consider an optimal hierarchy $H \in \Omega(N)$. Let k be the maximal number of employees immediately subordinated to common manager. If k=2 then the required optimal 2-hierarchy is H. If k>2 then consider a manager m with k immediately subordinated employees $v_1,\ldots,v_k$. Let $s_1=s_H(v_1),\ldots,s_k=s_H(v_k)$ be the groups controlled by the employees $v_1,\ldots,v_k$. As the cost function is narrowing there exist a number of employees $1 < j < k$ and permutation $(i_1,\ldots,i_k)$ satisfying inequality (35). Let's reconstruct the hierarchy H: hire new manager m_1 and immediately subordinate the employees $v_{i_1},\ldots,v_{i_j}$ to m_1 instead of m, immediately subordinate m_1 to m (see the example in Figure 29). Inequality (35) implies that the hierarchy cost does not increase. Thus, the obtained hierarchy is optimal. The manager m_1 has $j<k$ immediate subordinates. The manager m has $k - j + 1 < k$ immediate subordinates. So, in the obtained hierarchy the number of managers with k immediate subordinates decreases by one. We can repeat such reconstruction while there exists the manager with k immediate subordinates. As a result, we obtain the optimal hierarchy with maximal number $k' < k$ of employees immediately subordinated to common manager. If $k' > 2$ then we can repeat reconstructions.

As a result we obtain the optimal 2-hierarchy H_1. Proposition 1 implies that there exists 2-hierarchy H^* satisfying conditions (i)-(iii) (see page 24). Moreover, the cost of H^* is less than or equal to the cost of H_1. Thus, H^* is an optimal 2-hierarchy satisfying conditions (i)-(iii). ■

Proof of corollary (from Propositions 6 and 7). Proposition 6 implies that there exists an optimal tree because the cost function is group-monotonic. In the proof of Proposition 7 we can consider this tree as initial optimal hierarchy H. Lemma 2 (see page 17) implies that immediate subordinates of any manager control non-overlapping groups of workers. Therefore, there are no overlapping groups among the groups $s_1,\ldots,s_k$ in the proof of Proposition 7. So, we can reconstruct the hierarchy because the cost function is narrowing on non-overlapping groups. After the reconstruction (see proof of Proposition 7) we obtain some tree (new manager and each of other employees except the top manager have exactly one immediate superior). As a result, we obtain the optimal 2-tree. Similarly with the proof of Proposition 7 we can obtain the optimal 2-tree satisfying conditions (i)-(iii) of Proposition 1. ■

Proof of Proposition 8. Proposition 1 implies that there exists an optimal hierarchy $H \in \Omega(N)$, which satisfies conditions (i)-(iii) (see page 24). According to condition (ii) there exists manager m controlling all other employees.

If m is a single manager in the hierarchy then H is an optimal two-tier hierarchy. Otherwise there exists a manager m_1 immediately subordinated to the manager m. Let $v_1,\dots,v_j$ be all immediate subordinates of the managers m_1. Let $s_1=s_H(v_1),\dots,s_j=s_H(v_j)$ be the groups controlled by the employees $v_1,\dots,v_j$. As the hierarchy H satisfies condition (i) of Proposition 1 each manager has at least two immediate subordinates. So, $j>1$ and the manager m has other immediate subordinates besides m_1. Let $v_{j+1},\dots,v_k$, $k \geq 3$ be all such immediate subordinates. Let $s_{j+1}=s_H(v_{j+1}),\dots,s_k=s_H(v_k)$ be the groups controlled by the employees $v_{j+1},\dots,v_k$.

Suppose the manager m_1 has some immediate superiors m' besides m. So, there exist two different paths from m_1 to m: the first path contains only two nodes m_1 and m, the second path contains the manager m'. Besides m_1 the second path contains one of the managers immediately subordinated to m. Thus, the second path contains one of the employees $v_{j+1},\dots,v_k$. So, this employee controls the manager m_1. It contradicts condition (iii) of Proposition 1 (immediate subordinates of common manager do not control each other). Therefore, top manager m is the single immediate superior of the manager m_1.

Condition (iii) of Proposition 1 implies that there are no immediate subordinates of the manager m among the employees $v_1,\dots,v_j$ (otherwise immediate subordinate m_1 controls other immediate subordinate). So, there are no identical employees among $v_{j+1},\dots,v_k$ and $v_1,\dots,v_j$. Thus, the described fragment of the hierarchy looks like the fragment in Figure 29 b).

Inequality (36) holds for any groups $s_1,\dots,s_k$, $k \geq 3$, any number $1 < j < k$ and any permutation $(i_1,\dots,i_k)$ because the cost function is widening. If $(i_1,\dots,i_k)=(1,\dots,k)$ then inequality (36) is given by:

$$c(s_1,\dots,s_k) \leq c(s_1,\dots,s_j) + c(s_1 \cup \dots \cup s_j, s_{j+1},\dots,s_k). \qquad (*)$$

Let's reconstruct the hierarchy: immediately subordinate the employees $v_1,\dots,v_j$ to the manager m instead of the manager m_1 and remove the manager m_1. Obtained fragment of the graph looks like the fragment in Figure 29 a). The manager m controls all workers as before the reconstruction. So, the obtained graph is a hierarchy. The groups controlled by other managers do not change too. In the obtained hierarchy the cost of the manger m (first member of the inequality (*)) is less than or equal to costs of the managers m and m_1 in the initial hierarchy (right-hand member of inequality (*)). Thus, the obtained hierarchy is optimal.

The obtained hierarchy satisfies conditions (i) and (ii) of Proposition 1. But condition (iii) may be violated because some of the employees $v_1,\dots,v_j$ may be subordinated to some of the employees $v_{j+1},\dots,v_k$. Suppose the employee v_{j_1} is subordinated to the employee v_{j_2}, $1 \leq j_1 \leq j$, $j+1 \leq j_2 \leq k$. Lemma 1 (see page 16) leads to $s_{j_1} \subseteq s_{j_2}$. Lemma 4 (see page 23) implies that "excess" edge (v_{j_1}, m) can be removed with no hierarchy cost increase. After removal the employee v_{j_1} is subordinated to the top manager but not immediately (through the employee v_{j_2}). We can repeat such removal. As a result, we obtain the optimal hierarchy satisfying conditions (i), (ii) and (iii) of Proposition 1.

The obtained optimal hierarchy contains less managers then the initial hierarchy because the manager m_1 has been removed. We can repeat similarly reconstructions while there are two or more managers in the hierarchy. As a result, we obtain the optimal two-tier hierarchy with the single manager m. ■

Proof of corollary (from Propositions 6 and 8). Proposition 6 implies that there exists an optimal tree because the cost function is group-monotonic. In the proof of Proposition 8 we can consider this tree as initial optimal hierarchy H. Lemma 2 (see page 17) implies that immediate subordinates of any manager control non-overlapping groups of workers. Therefore, there are no overlapping groups among the groups $s_1,\ldots,s_k$ in the proof of Proposition 8. So, we can reconstruct the hierarchy because the cost function is widening on non-overlapping groups. After the reconstruction (see proof of Proposition 8) we obtain some tree (each of employees except the top manager has exactly one immediate superior). As a result, we obtain the optimal two-tier hierarchy. ■

Proof of Proposition 9. Proposition 7 implies that there exists an optimal 2-hierarchy H_1 because the cost function is narrowing. Proposition 1 implies that there exists an optimal 2-hierarchy H satisfying conditions (i)-(iii) (see page 24). Below in the proof condition (i) will be used. So, all employees in the hierarchy control different groups of workers. Particularly, each manager has exactly two immediate subordinates.

A manager will be called *incorrect* if he or she has two immediately subordinated managers. If there are no incorrect managers in H then each manager has at least one immediately subordinated worker. In this case H is an optimal consecutive hierarchy. If there are incorrect managers in H then we will decrease the number of such managers because of reconstruction with no hierarchy cost increase.

Consider an incorrect manager m which controls only correct managers. m has two immediately subordinated managers m_1 and m_2. Correct manager m_1 immediately controls the worker w' and the employee v'. Correct manager m_2 immediately controls the worker w'' and the employee v''. Corresponding fragment of the hierarchy looks like the fragment shown in Figure 33 a).

Let $s_1=s_H(m_1)$ and $s_2=s_H(m_2)$ be the groups controlled by the managers m_1 and m_2. Condition (i) of Proposition 1 implies that the employee v' cannot control the worker w' because in this case m_1 and v' controls the same group. So, $s_H(v') = s_1 \setminus \{w'\}$. Similarly $s_H(v'') = s_2 \setminus \{w''\}$.

If the groups s_1 and s_2 satisfy the condition a) of Definition 11 then we can reconstruct the hierarchy with no cost increase. Let's hire new manager m_3 and immediately subordinate employees v' and m_2 to m_3. Let's immediately subordinate the worker w' and the manager m_3 to the manager *m*. The fragment of obtained hierarchy is shown in Figure 33b).

New manager m_3 controls the group $(s_1 \setminus \{w'\}) \cup s_2$. Before the reconstruction the cost of the manager *m* equals $c(s_1,s_2)$. After the reconstruction manager's m_3 cost adds to the cost of the hierarchy. But manager's *m* cost decreases. Thus, the difference between the cost of initial hierarchy and the cost of obtained hierarchy equals $c(s_1, s_2) - c(s_1 \setminus \{w'\}, s_2) - c((s_1 \setminus \{w'\}) \cup s_2, \{w'\}) \geq 0$. So, the cost of hierarchy does not increase. Therefore, the obtained hierarchy is optimal.

If the groups s_1 and s_2 satisfies the condition b) of Definition 11 then the hierarchy can be reconstructed similarly: the worker w'' is immediately subordinated to the manager *m*. The fragment of obtained hierarchy is shown in Figure 33c).

So, if the cost function is strongly narrowing then we can construct the optimal 2-hierarchy with correct manager *m*. In the obtained hierarchy condition (i) of Proposition 1 may be violated because the manager m_3 and some other manager m' may control the same group. In this case we can immediately subordinate m' to *m* and remove the manager m_3.[80] The obtained hierarchy is optimal and satisfies condition (i). The manager *m* is correct in the obtained hierarchy.

If m_3 is a correct manager (or m_3 has been removed) then the number of incorrect managers in obtained hierarchy is less than the number in initial hierarchy *H*. Suppose the obtained hierarchy contains the incorrect manager m_3. *m* controls wider group of workers than m_3. So, the new incorrect manager m_3 controls smaller group than the initial manager *m*. We can repeat the reconstruction with the manager m_3 instead of *m*. The number of workers controlled by the incorrect managers decreases after each reconstruction. As a result, we obtain the optimal hierarchy with less number of incorrect managers than the initial hierarchy *H*.

[80] *m* is an only immediate superior of the manager m_3. Therefore, m_3 removal does not change the groups controlled by the managers in the hierarchy. So, the costs of these managers do not change too.

We can repeat described above reconstructions while there exist incorrect managers. As a result, we obtain the optimal hierarchy H_2 without incorrect managers. The optimal hierarchy H_2 is consecutive.

After Definition 10 we show that Proposition 1 is true for consecutive hierarchies. So, there exists the consecutive hierarchy H^* satisfying conditions (i)-(iii) of Proposition 1 (see page 24). And the cost of the hierarchy H^* is less than or equal to the cost of the hierarchy H_2. So, H^* is optimal consecutive hierarchy (see Figure 32). ■

Proof of Proposition 10. Consider a groups $s_1,\dots,s_k$, $k \geq 3$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequalities (35), (36) (see Section 3.3). The inequalities correspond with narrowing and widening cost functions (see Definition 9 on page 87).

Suppose $\beta \leq 1$. Let's prove inequality (36) for any $1 < j < k$ and any permutation $(i_1,\dots,i_k)$. Inequality (36) is given by $c(s_1,\dots,s_k) \leq c(s_{i_1},\dots,s_{i_j}) + c(s_{i_1} \cup \dots \cup s_{i_j}, s_{i_{j+1}},\dots,s_{i_k})$. Let's define the following values: $x_1 = \mu(s_{i_1})^\alpha,\dots,x_j = \mu(s_{i_j})^\alpha$, $x' = \max(x_1,\dots,x_j)$, $x = x_1+\dots+x_j$, $y_{j+1} = \mu(s_{i_{j+1}})^\alpha, y_{j+2} = \mu(s_{i_{j+2}})^\alpha,\dots,y_k = \mu(s_{i_k})^\alpha$, $y' = \max(y_{j+1},\dots,y_k)$, $y = y_{j+1}+\dots+y_k$, $s = s_{i_1} \cup \dots \cup s_{i_j}$. Then the left-hand member and the right-hand member of inequality (36) are given by:

$$z_1 = (x + y - \max(x', y'))^\beta,$$

$$z_2 = (x - x')^\beta + (\mu(s)^\alpha + y - \max(y', \mu(s)^\alpha))^\beta.$$

Inequality (38) and $\beta \leq 1$ imply that the inequality $z_2 \geq (x + y + \mu(s)^\alpha - x' - \max(y', \mu(s)))^\beta$ holds. To prove inequality (36) ($z_2 \geq z_1$) it is enough to prove:

$$x + y - \max(x', y') \leq x + y + \mu(s)^\alpha - x' - \max(y', \mu(s)^\alpha).$$

This inequality is given by: $x' + \max(y', \mu(s)^\alpha) \leq \mu(s)^\alpha + \max(x', y')$. If $y' \leq \mu(s)^\alpha$ then the inequality is simplified: $x' \leq \max(x', y')$. So, the inequality holds. If $y' > \mu(s)^\alpha$ then the inequality is given by $x' + y' \leq \mu(s)^\alpha + \max(x', y')$. The inequalities $y' \leq \max(x', y')$ and $x' \leq \mu(s)^\alpha$ hold because $s = s_{i_1} \cup \dots \cup s_{i_j}$. Thus, inequality (36) holds. So, if $\beta \leq 1$ then function (I) is widening.

Suppose $\beta \geq 1$. Let s_1 be the group with maximal complexity: $\mu(s_1) = \max(\mu(s_1), \dots, \mu(s_k))$ (otherwise we can renumber the groups $s_1, \dots, s_k$,). Consider the groups s_1, s_2 (j=2) and the permutation $(1,2,\dots,k)$. Let's prove inequality (35) which is given by: $c(s_1, \dots, s_k) \geq c(s_1, s_2) + c(s_1 \cup s_2, s_3, ..., s_k)$. The left-hand member and the right-hand member of inequality (35) are given by: $z_1 = (\mu(s_2)^\alpha + \dots + \mu(s_k)^\alpha)^\beta$, $z_2 = \mu(s_2)^{\alpha\beta} + (\mu(s_3)^\alpha + \dots + \mu(s_k)^\alpha)^\beta$. Inequality (37) and $\beta \geq 1$ lead to $z_1 \geq z_2$. Thus, inequality (35) holds. So, if $\beta \geq 1$ then function (I) is narrowing.

Suppose $\alpha\beta \geq 1$ and $\beta \geq 1$. Let's prove that function (I) is strongly narrowing (see Definition 11 on page 98) on this set of parameters. Let s_1 and s_2 be any groups with two or more workers in each group. Consider the case $\mu(s_1) \leq \mu(s_2)$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequality a) of Definition 11:

$$z_1 = c(s_1, s_2),\ z_2 = c(s_1 \setminus \{w\}, s_2) + c((s_1 \setminus \{w\}) \cup s_2, \{w\}),$$

where w is any worker from the group s_1. Let's define the following values: $x = \mu(s_1)$, $y = \mu(s_1 \setminus \{w\})$, $z = \mu(\{w\})$. So, $z_1 = x^{\alpha\beta}$, $z_2 = y^{\alpha\beta} + z^{\alpha\beta}$, x=y+z. Then inequality a) of Definition 11 ($z_1 \geq z_2$) is given by $(y+z)^{\alpha\beta} \geq y^{\alpha\beta} + z^{\alpha\beta}$. Inequality (37) and $\alpha\beta \geq 1$ lead to $z_1 \geq z_2$.

If $\mu(s_1) \geq \mu(s_2)$ then inequality b) of Definition 11 can be proved similarly (s_1 and s_2 replace each other).

Therefore, if $\beta \geq 1$ and $\alpha\beta \geq 1$ then function (I) is strongly narrowing. ■

Proof of Proposition 11. Consider a groups $s_1, \dots, s_k$, $k \geq 3$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequality (36) (see page 88). The inequality corresponds with widening cost function (see Definition 9).

Let's prove inequality (36) for any $1 < j < k$ and any permutation $(i_1, \dots, i_k)$. Inequality (36) is given by:

$$c(s_1, \dots, s_k) \leq c(s_{i_1}, \dots, s_{i_j}) + c(s_{i_1} \cup \dots \cup s_{i_j}, s_{i_{j+1}}, ..., s_{i_k}).$$

Let's define the following values: $s = s_{i_1} \cup \ldots \cup s_{i_j}$, $x = \mu(s_{i_1})^{\alpha} + \ldots + \mu(s_{i_j})^{\alpha}$, $y = \mu(s_{i_{j+1}})^{\alpha} + \ldots + \mu(s_{i_k})^{\alpha}$. Then the left-hand member and the right-hand member of inequality (36) are given by: $z_1 = (x+y)^{\beta}$, $z_2 = x^{\beta} + (\mu(s)^{\alpha} + y)^{\beta}$.

If $\beta \le 1$ then (38) leads to $z_1 \le x^{\beta} + y^{\beta} \le z_2$. Thus, inequality (36) holds. So, if $\beta \le 1$ then function (II) is widening.

If the groups $s_1,\ldots,s_k$ are non-overlapping then $\mu(s) = \mu(s_{i_1}) + \ldots + \mu(s_{i_j})$. If $\alpha \ge 1$ then (37) leads to $\mu(s)^{\alpha} \ge \mu(s_{i_1})^{\alpha} + \ldots + \mu(s_{i_j})^{\alpha} = x$. Thus, the inequality $z_2 \ge (x+y)^{\beta} = z_1$ (inequality (36)) holds too. So, if $\beta > 1$ and $\alpha \ge 1$ then function (II) is widening on non-overlapping groups.

If the groups $s_1,\ldots,s_k$ are overlapping then inequality (36) may be violated. For example, $s_1=\{w_1,w_2\}$, $s_2=\{w_1,w_3\},\ldots,s_{k-1}=\{w_1,w_k\}$, $s_k=\{w_{k+1}\}$, $\mu(w_1) = \mu(w_{k+1}) = 1$, $\mu(w_2) = \ldots = \mu(w_k) = 0$. Consider the number $j=k-1$ and the identity permutation $(1,\ldots,k)$. Then $x=k-1$, $y=1$, $\mu(s)^{\alpha} = 1$. The left-hand member and the right-hand member of inequality (36) are given by $z_1 = k^{\beta}$ and $z_2 = (k-1)^{\beta} + 2^{\beta}$. Inequality (36) is given by $z_1 - z_2 \le 0$ or $k^{\beta} - (k-1)^{\beta} \le 2^{\beta}$. For any $\beta > 1$ the left-hand member increases with k. If k is large enough then inequality (36) is violated. So, if $\beta > 1$ and $\alpha \ge 1$ function (II) is widening only on non-overlapping groups.

Let's prove that if $\beta > 1$ and $\alpha < 1$ then function (II) is neither widening nor narrowing. Let's prove it on non-overlapping groups.

Consider the groups $s_1=\{w_1\}, s_2=\{w_2\},\ldots,s_k=\{w_k\}$, $\mu(w_1) = \ldots = \mu(w_k) = 1$, where $k > 1$ is an even number. Consider the number $j=k/2$ and identity permutation $(1,\ldots,k)$. Then $x=k/2$, $y=k/2$, $\mu(s)^{\alpha} = (k/2)^{\alpha}$. The left-hand member and the right-hand member of inequality (36) are given by $z_1 = k^{\beta}$, $z_2 = (k/2)^{\beta} + ((k/2)^{\alpha} + k/2)^{\beta}$. Inequality (36) is given by $z_2 \le z_1$ or $\left(\frac{1}{2}\right)^{\beta} + \left(\frac{1}{2^{\alpha} k^{1-\alpha}} + \frac{1}{2}\right)^{\beta} \ge 1$. If $\alpha < 1$ then the left-hand member decreases with k increase. If k is large enough then the value of the left-hand member is arbitrary closely to $1/2^{\beta-1}$. If $\beta > 1$ then $1/2^{\beta-1}$ is less than 1. Thus, if k is large enough then inequality (36) is violated. So, if $\beta > 1$ and $\alpha < 1$ then function (II) is not widening even on the non-overlapping groups.

Consider the groups $s_1=\{w_1\}$, $s_2=\{w_2\}$, $s_3=\{w_3\}$, $\mu(w_1)=\mu(w_2)=1$, $\mu(w_3)=0$. Let's prove that for any number $1<j<3$ (i.e. j=2) and any permutation (i_1,i_2,i_3) inequality (35) is violated. Inequality (35) is given by:

$$c(s_1,\ldots,s_k) \geq c(s_{i_1},\ldots,s_{i_j}) + c(s_{i_1} \cup \ldots \cup s_{i_j}, s_{i_{j+1}},\ldots,s_{i_k}) .$$

If the permutation equals (1,2,3) or (2,1,3) then inequality (35) is given by $2^\beta \geq 2^\beta + 2^{\alpha\beta}$. For all other permutations inequality (35) is given by $2^\beta \geq 1+2^\beta$. Thus, there does not exist number $1<j<3$ and permutation (i_1,i_2,i_3) satisfying inequality (35). So, function (II) is not widening even on the non-overlapping groups. ■

Proof of Proposition 12. Suppose $\beta \geq 1$. At first let's prove that function (III) is narrowing. Consider a groups $s_1,\ldots,s_k$, $k \geq 3$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequality (35) (see page 88). The inequality corresponds with narrowing cost function (see Definition 9). Let s_1 be the group with maximal complexity: $\mu(s_1)=\max(\mu(s_1),\ldots,\mu(s_k))$ (otherwise we can renumber the groups $s_1,\ldots,s_k$). Consider the groups s_1, s_2 (j=2) and the permutation $(1,2,\ldots,k)$. Let's prove inequality (35) which is given by: $c(s_1,\ldots,s_k) \geq c(s_1,s_2) + c(s_1 \cup s_2, s_3,\ldots,s_k)$.

Let's define the values $x=\mu(s_1 \cup \ldots \cup s_k)^\alpha$, $y=\mu(s_1 \cup s_2)^\alpha$, $z=\mu(s_1)^\alpha$. Then $z \leq y \leq x$. The left-hand member and the right-hand member of inequality (35) are given by $z_1=(x/z-1)^\beta$, $z_2=(y/z-1)^\beta+(x/y-1)^\beta$.[81] Inequality (37) and $\beta \geq 1$ imply that the inequality $z_2 \leq (y/z-1+x/y-1)^\beta$ holds. Using this estimation we can prove inequality (35) $(z_2 \leq z_1)$ with the help of proving the inequality $x/z-1-y/z+1-x/y+1 \geq 0$. This inequality is given by:

[81] If z=0 then $z_1=+\infty$. So, the inequality $z_1 \geq z_2$ holds. Below we suppose $z>0$.

$$(xy + yz - y^2 - xz)/yz = (x - y)(y - z)/yz \geq 0 .$$

Thus, inequality (35) holds. So, if $\beta \geq 1$ then function (III) is narrowing.

Let's prove that function (III) is strongly narrowing (see Definition 11 on page 98). Let s_1 and s_2 be any groups with two or more workers in each group. Consider the case $\mu(s_1) \leq \mu(s_2)$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequality a) of Definition 11:

$$z_1 = c(s_1, s_2),\ z_2 = c(s_1 \setminus \{w\}, s_2) + c((s_1 \setminus \{w\}) \cup s_2, \{w\}),$$

where w is any worker from the group s_1. Let's define the following values: $x = \mu(s_1 \cup s_2)^{\alpha}$, $y = \mu((s_1 \setminus \{w\}) \cup s_2)^{\alpha}$, $z = \mu(s_2)^{\alpha}$. So, $z \leq y \leq x$, $z_1 = (x/z - 1)^{\beta}$, $z_2 = (y/z - 1)^{\beta} + (x/y - 1)^{\beta}$. The following inequality $(x/z - 1)^{\beta} \geq (y/z - 1)^{\beta} + (x/y - 1)^{\beta}$ was proven above. Thus, the inequality $z_1 \geq z_2$ holds.

If $\mu(s_1) \geq \mu(s_2)$ then inequality b) of Definition 11 can be proved similarly by replacing s_1 by s_2 and vice versa.

Therefore, if $\beta \geq 1$ then function (III) is strongly narrowing. ■

Proof of Proposition 13. Suppose $\beta \geq 1$. Let's prove that function (IV) is narrowing. Consider a groups $s_1,\ldots,s_k$, $k \geq 3$. Let z_1 and z_2 be the left-hand member and the right-hand member of inequality (35) (see page 88). The inequality corresponds with narrowing cost function (see Definition 9). Consider the groups s_1, s_2 (j=2) and the permutation $(1,2,\ldots,k)$. Let's prove inequality (35) which is given by:

$$c(s_1,\ldots,s_k) \geq c(s_1, s_2) + c(s_1 \cup s_2, s_3,\ldots,s_k).$$

Let's define the values $x = \mu(s_1 \cup \ldots \cup s_k)^{\alpha}$, $y = \mu(s_1 \cup s_2)^{\alpha}$, $x_1 = \mu(s_1)^{\alpha},\ldots, x_k = \mu(s_k)^{\alpha}$. The left-hand member and the right-hand member of inequality (35) are given by $z_1 = (kx - x_1 - \ldots - x_k)^{\beta}$, $z_2 = (2y - x_1 - x_2)^{\beta} + ((k-1)x - y - x_3 - \ldots - x_k)^{\beta}$. Inequality (37) and $\beta \geq 1$ imply that the inequality $z_2 \leq ((k-1)x + y - x_1 - \ldots - x_k)^{\beta}$ holds. The right-hand member is less than or equal to z_1 because $y \leq x$. Thus, inequality (35) ($z_2 \leq z_1$) holds. So, if $\beta \geq 1$ then function (IV) is narrowing. ■

Proof of Proposition 14. The equality $\gamma=\alpha-\beta$ holds for function (V). Let's substitute expression (40) (function (V) for non-overlapping groups) in expression (39). Then the cost of infinite tree is given by:

$$x^{\gamma}(y_1+\ldots+y_k)^{\alpha}/[\min(y_1^{\beta},\ldots,y_k^{\beta})(1-\sum_{i=\overline{1,k}} y_i^{\gamma})]. \qquad (*)$$

The numerator in the brackets equals 1 ($y_1+\ldots+y_k=1$). To minimize the expression it is enough to maximize the denominator. It is obvious that the expression $\min(y_1^{\beta},\ldots,y_k^{\beta})$ reaches maximum when $y_1=\ldots=y_k=1/k$.

With the help of the simplest mathematical analysis methods we can prove that for $\gamma>1$ the expression $(1-\sum_{i=\overline{1,k}} y_i^{\gamma})$ reaches maximum when $y_1=\ldots=y_k=1/k$.

Thus, the symmetric k-tree minimizes cost function (V). In this tree each manager has exactly k immediate subordinates. These subordinates control the groups with the same complexity. So, we have to find optimal k. Without the constant x^{γ} expression (*) with $y_1=\ldots=y_k=1/k$ is given by the function $\xi(k)$:

$$\xi(k)=k^{\beta}/(1-k/k^{\gamma})=k^{\beta+\gamma-1}/(k^{\gamma-1}-1)=k^{\alpha-1}/(k^{\alpha-\beta-1}-1).$$

Let's differentiate the function by k and ignore the positive multiplier:

$$\xi'(k)=(\alpha-1)k^{\alpha-2}(k^{\alpha-\beta-1}-1)-(\alpha-\beta-1)k^{\alpha-\beta-2}k^{\alpha-1}=$$
$$=k^{\alpha-2}[(\alpha-1)(k^{\alpha-\beta-1}-1)-(\alpha-\beta-1)k^{\alpha-\beta-1}]=k^{\alpha-2}[\beta k^{\alpha-\beta-1}-(\alpha-1)].$$

The sign of the derivative depends only on the sign of the expression in the brackets. The derivative equals to zero when $k=r_0=((\alpha-1)/\beta)^{1/(\alpha-\beta-1)}$. If $k<r_0$ then the derivative is negative (the cost of the tree decreases) because of $\alpha-\beta-1>0$. If $k>r_0$ then the derivative is positive (the cost of the tree increases). Thus, r_0 is minimal point. If r_0 is not an integer then one of the nearest two integers is minimal point (the maximal integer is less than r_0 or the minimal integer is greater than r_0). To obtain the minimal point it is enough to compare the values of $\xi(k)$ in these two points. ■

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