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OPTIMAL ORGANIZATIONAL HIERARCHIES FOR HOMOGENOUS MANAGER COST FUNCTION

We consider a problem of organizational structure design as a problem of discrete optimization, i.e., that of the search for an admissible hierarchy minimizing total managers' maintenance costs. The problem is solved for the case of the, so-called, homogeneous manager cost function. It was proved that each manager in the optimal hierarchy has an approximately equal span of control (number of immediate subordinates) and divides the subordinate department among them in a roughly identical proportion. A closed-form expression is deduced for the lower-bound estimate of the optimal hierarchy cost. This lower bound has many applications and, in many cases, allows constructing nearly optimal organizational hierarchies.

1. Introduction

Optimal hierarchy[1] problems are met in various applications. For instance, in the course of organizational design (i.e., construction of a rational organization chart) a management hierarchy is built over a set of lower-level production workers determined by manufacturing technology [2] of the firm. Similar problems arise in production planning and scheduling, design of a quality control system, and development of data collection structures. Many discrete optimization problems reduce to the search of optimal hierarchy (e.g., the classic problem of the optimal prefix code [3]).

Firstly, we have a collection of admissible hierarchies built over a fixed set of lower-level elements. Secondly, we have an efficiency criterion, which allows comparing admissible hierarchies. This criterion should be minimized or maximized by the choice of an admissible hierarchy. A typical criterion used in organizational design is the value or the profit of the firm, which is maximized, or management expenses, i.e., management hierarchy maintenance costs, which are minimized. In the present paper we minimize the cost of the hierarchy.

[1] *Hierarchy* is a principle of organization of complex multilevel systems, which bases on the ordering of levels from higher to lower [**Ошибка! Источник ссылки не найден.**, P. 201]. Such an ordering is typically induced by relations of authority or subordination.

The shape of organizational structure is widely agreed to be an important factor of management efficiency and, consequently, of organizational success (e.g., see [4-6]). Yet, nowadays there is still lack of general and universally recognized theories of rational organizational structure design. Below we sketch a general model of hierarchy optimization and study in detail the important special case of homogeneous manager costs.

2. General model of hierarchy optimization[2]

Hierarchical structures of complex systems are typically modeled with acyclic directed graphs [11]. Let $H = <V, E>$ be a *directed graph* with the vertex set V and the arc set $E \subseteq V \times V$. If a vertex pair (v_1, v_2) belongs to the arc set E, an arc is drawn from vertex v_1 to vertex v_2. The graph $H = <V, E>$ is called *acyclic* if one cannot return to a starting vertex moving along graph arcs.

An organizational hierarchy coordinates activity of a fixed set of workers (lower-level employees). The manufacturing technology of a firm determines some set of workers, and it is over this set of workers a hierarchy of supervisors (or managers) is built. It is a traditional approach for most mathematical models of organizational hierarchy design to distinguish workers ("doers") and managers ("rulers"). Let $N = \{w_1, \ldots, w_n\}$ be the fixed *set of workers* ($n > 1$), and M be a finite set of *managers* controlling these workers (the set M may change from hierarchy to hierarchy).

An *organizational hierarchy* is an acyclic graph $H = <V, E>$ with the vertex set $V = N \cup M$ and the arc set $E \subseteq V \times M$. Elements of the set V will be referred to as *employees*. Graph arcs reflect subordination: if an arc presents in G from employee v_1 to employee v_2, then employee v_1 is *immediately subordinated* to employee v_2 in the hierarchy H, and employee v_2 is an immediate superior (a boss) of the employee v_1. Therefore, arcs are directed from the subordinate to his immediate superior.[3] If a chain $v_1, \ldots, v_k$ of employees presents in G, such that $(v_i, v_{i+1}) \in E$ for $i = 1, \ldots, k - 1$, we say that employee v_1 is *subordinated* to employee v_k, or that employee v_k *controls* employee v_1. Taking care of the other natural properties of organizational structures we come to the following formal definition.

Definition 1 [8]. *A directed acyclic graph* $H =< N \cup M, E >$ *with the arc set* $E \subseteq (N \cup M) \times M$ *is called a* ***hierarchy*** *over the set of workers N, if any manager from the set M has at least one subordinate, and there exists a manager (a top-manager) controlling all workers from N. A collection of all organizational hierarchies over the set of workers N is denoted with* $\Omega(N)$.

The above definition is illustrated with Fig. 1. Workers are depicted with the dark circles and labeled with Arabic digits, while managers are depicted with light circles and labeled with Roman digits. Graphs (a)-(c) are organizational hierarchies over the set of workers $N = \{1, \ldots, 4\}$.

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[2] This model is based on the approach and notation suggested by A. Voronin and S. Mishin [7-**Ошибка! Источник ссылки не найден.**]. The present paper is merely a further development of their findings.

[3] It is clear that workers have no subordinates, and all arcs of an organizational hierarchy are directed towards managers.

They illustrate elements of organizational structures often met in management practice: inter-level interaction, when both managers and workers are subordinated to a single manager (e.g. manager II in the hierarchy (b)) and multiple subordination (common agency) when an employee has more than one immediate superior (e.g. manager I in the hierarchy (b) or worker 3 in the hierarchy (c)). At the same time, graphs (d)-(f) are **not** organizational hierarchies. Employee 3 has subordinates in the graph (d), the graph (e) has no top-manager controlling all workers, manager II in the graph (f) has no subordinates, and graph (f) has the cycle 1→I→III→1.

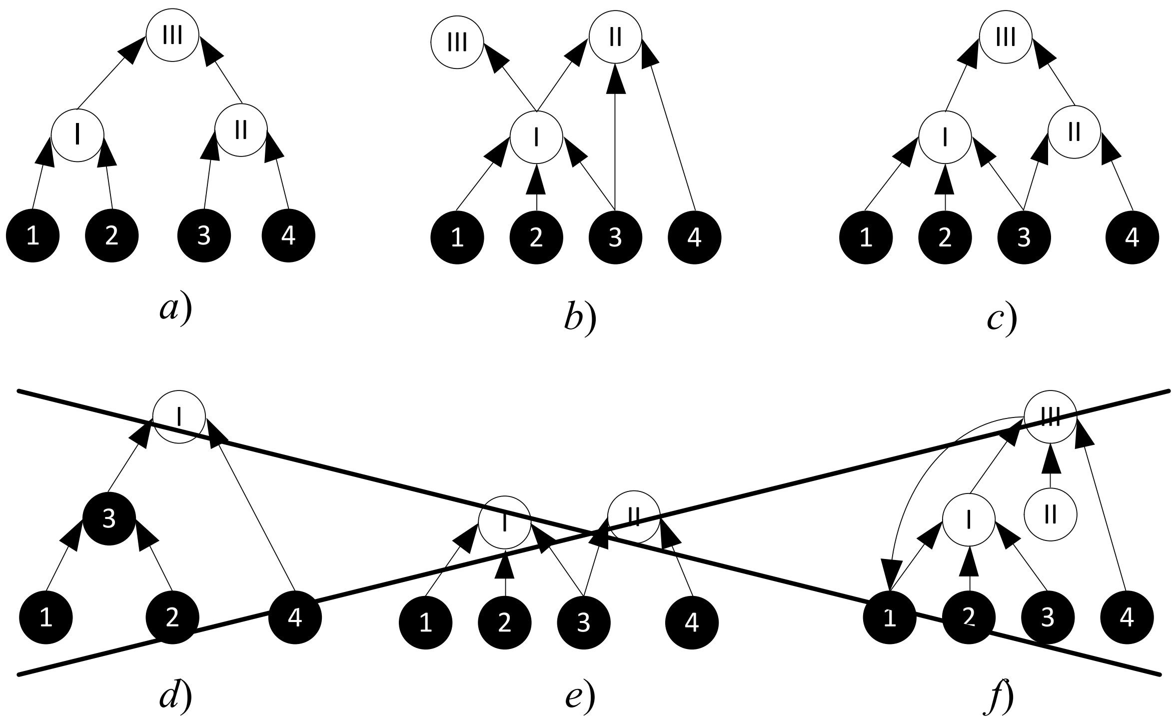

Fig. 1. To the definition of organizational hierarchy

Let us introduce several typical hierarchy shapes.

Definition 2 [8]. *A hierarchy H is called a* ***tree*** *when all employees except a top-manager have a single immediate superior, while a top-manager has no superiors.*

Definition 3 [8]. *Fix a natural number* $r > 1$. *A hierarchy H is called an* ***r-hierarchy*** *if every manager in H has no more than r immediate subordinates. If an r-hierarchy is a tree, it is called an* ***r-tree***.

Definition 4 [8]. *A* ***fan*** *hierarchy is a hierarchy with a sole manager, who immediately controls all workers.*

A general optimal hierarchy problem is set a follows. Let a set of workers N is given along with a collection $\Omega \subseteq \Omega(N)$ of admissible hierarchies, and a cost function $C(H)$ assigns a non-negative number to each admissible hierarchy $H \in \Omega$. The problem is to find an admissible hierarchy with minimum cost, i.e., to find

$$H^* \in \operatorname{Arg}\min_{H \in \Omega} C(H).$$

The collection Ω of admissible hierarchies either coincides with the collection $\Omega(N)$ of all hierarchies over the set of workers N or constitutes its strict subset. In particular, depending on the problem in hand an optimal tree or an optimal r-hierarchy can be searched for. Many models of optimal organizational hierarchy from the literature reduce to the above setting (see [4, 6, 11-19].)[4]

The cost function $C(H)$ reflects the expenses connected with maintenance of the management hierarchy H in the organization. Below we assume that the cost $C(H)$ of an arbitrary hierarchy H can be decomposed into the sum of managers' maintenance costs. In other words, we can write the hierarchy cost as $C(H)=\sum_{m\in M} c(m,H)$, where a non-negative function $c(m, H)$ (the, so-called, manager cost function) collects all expenditures connected with maintenance of manager m in the hierarchy H.

3. Homogeneous cost functions

Efficient methods for optimal hierarchy search can be developed only for special classes of the manager cost function. Below we introduce the class of *homogeneous* cost functions, which address a variety of optimal hierarchy problems. Let us start with defining the auxiliary concepts.

Any non-empty subset $s \subseteq N$ of a set of workers N is called a *group* of *workers*. For any manager $m \in M$ in the hierarchy H define a *subordinate group of workers* $s_H(m) \subseteq N$, i.e., the group of workers being subordinated to manager m in the hierarchy H. We also say that manager m *controls* the group of workers $s_H(m)$. It will be convenient to think that a worker $w \in N$ controls the group $\{w\}$, which consists of this worker himself.

Definition 5 [9]. *The manager cost function is called* ***sectional*** *if its value for a manager m in a hierarchy H depends only on the groups of workers controlled by immediate subordinates of manager m.*

Therefore, if r employees $v_1, v_2,\ldots, v_r$ are immediately subordinated to manager m in the hierarchy H, the cost of manager m in the hierarchy H is written as

$$c(m, H) = c(s_H(v_1), \ldots, s_H(v_r)).$$

In general, immediate subordinates of a manager may control intersecting groups of workers. For example, manager m in Fig. 2 has three immediate subordinates (manager m_1, manager m_2, and worker w_8) and her cost is written as: $c(m,H)=c(\{w_1,w_2,w_3,w_4\},\{w_3,w_4,w_5,w_6,w_7\},\{w_8\})$.

The argument of any sectional cost function $c(s_1, \ldots, s_r)$ is a collection of groups of workers, and, hence, a sectional function is not easy to study. Just to define a specific sectional cost function one may, in general, require exhaustive enumeration of its values for all possible collections of groups of workers, which is practically impossible due to the enormous number of such collections. If we need a closed-form expression for a sectional cost function, we have to assign numeric

[4] In book [20] the general model adopted in the present paper is compared with many existing approaches to the optimal organizational hierarchy problem.

characteristics to any group of workers or a collection of groups, and to relate manager costs to these metrics rather than to groups of workers themselves.[5]

The simplest numeric characteristic is a *measure* defined on the set of workers. Each worker $w \in N$ is assigned a positive measure $\mu(w)$. The measure $\mu(s)$ of the group of workers $s \subseteq N$ is equal to the sum of measures of its members. Then we assume that the manager cost function can be written as the following function of $r + 1$ variables: $c(s_1, \ldots, s_r) = c(\mu_1, \ldots, \mu_r, \mu)$, where $\mu_1, \ldots, \mu_r$ are the measures of groups controlled by the immediate subordinates of manager m, while μ is the measure of the group controlled by manager m herself. Such a cost function is called *measure-dependent* function.[6]

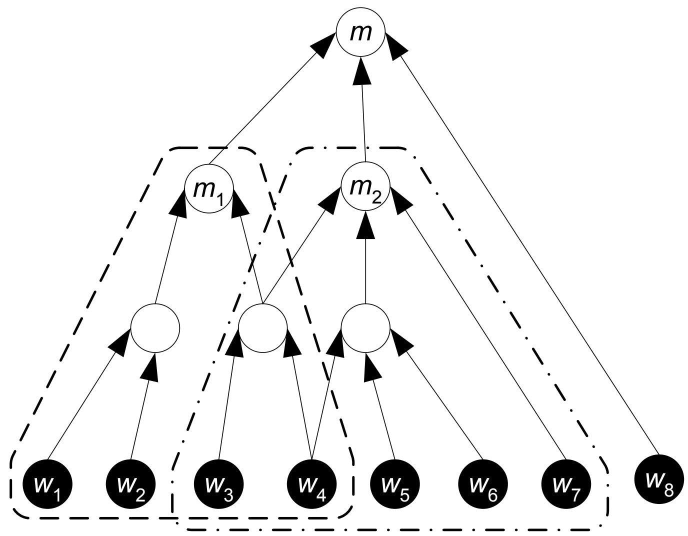

Fig. 2. Subordinate groups of workers

Consider a simplistic case of all workers' measures equal to unity. Then the measure of a group of workers reduces to its *cardinality*, the number of workers in the group, and the manager cost function depends on the total number of subordinate workers and on the number of workers controlled by each of immediate subordinates of the manager.

Example 1. The *multiplicative function* is an example of the measure-dependent cost function. Under a multiplicative cost function the manager's cost depends on the number of immediate subordinates r and on the measure μ of the subordinate group, and is written as $c(r, \mu) = \varphi(r)\chi(\mu)$, where $\varphi(\cdot)$ и $\chi(\cdot)$ are non-negative monotone functions. Under a multiplicative function the manager cost is obtained by scaling the cost $\varphi(r)$ of coordinating a team of r immediate subordinates to the total size μ of the part of organization under control using a scaling factor $\chi(\mu)$.

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[5] See [7, 8, 9, **Ошибка! Источник ссылки не найден.**] for some results on general sectional cost functions.

[6] Having in mind organizational design applications we can relate the worker's measure to the complexity of work he performs in an organization. Then the measure of a group of workers corresponds to the total complexity of group activities, and this value is supposed to determine the administrative cost of controlling this group of workers. Note that the manager cost function is defined for any number of her immediate subordinates r and is symmetric with respect to permutation of arguments $\mu_1, \ldots, \mu_r$ (but not the last argument μ).

Now we introduce the notion of homogeneous cost functions being the main object of the present study.

Definition 6 [8]. *A measure-dependent cost function $c(\mu_1, \ldots, \mu_r, \mu)$ is called* ***homogeneous*** *if such a non-negative number γ exists that for any positive number A and for any valid collection of group measures $\mu_1, \ldots, \mu_r, \mu$* the identity $c(A\mu_1,\ldots,A\mu_r,A\mu) = A^\gamma c(\mu_1,\ldots,\mu_r,\mu)$ *holds. The number γ is called the degree of homogeneity of the cost function.*

The value of a homogeneous function is multiplied by A^γ when all its arguments are multiplied by A. In particular, a multiplicative cost function $c(r, \mu) = \varphi(r)\chi(\mu)$ is homogeneous if and only if $\chi(\mu) = \mu^\gamma$.

There is some empiric background in the economical literature to consider manager costs in an organization as a homogeneous function (in the sense stated above). From the seminal work by Roberts [21] to the recent contributions (e.g., see [22]) a number of empiric studies (see [23-25]) justify the power relation between the executive's compensation in a firm and the size of this firm[7]. It is interesting to note that the relation $c = s^\gamma$ between the executive's compensation c and the size s of the company under control of this executive (being it gross profit, total asset cost, or some other metric) is surprisingly stable in time and in space, with only weak dependence on the industry and the location of the firm. The above considerations motivate our interest in studying homogeneous manager cost functions[8].

4. Problem setting and solution approach

In the present paper we solve the problem of the optimal tree-shaped hierarchy under a homogeneous manager cost function.[9] In the previous section the motivation is provided for the study of homogeneous cost functions, and, before we continue, several words should be said on the importance of tree-shaped hierarchies.

Firstly, there is a long tradition of modeling organizational structures as trees. In most models met in the literature [18, 27, 28 and others] a hierarchy is postulated to be a tree by definition. Secondly, many real-world organizational structures are tree-shaped (although, for instance, in matrix structures multiple subordination is allowed [29, **Ошибка! Источник ссылки не найден.**]). Thirdly, for many sectional cost functions multiple subordination is formally proved to be suboptimal. Mishin suggested a convenient criterion for a sectional manager cost function to assure some tree is optimal among all hierarchies over a given set of workers (see details in [8, 9]). In particular, if a measure-dependent manager cost function $c(\mu_1, \ldots, \mu_r, \mu)$ is monotone increasing

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[7] Executives' compensation is very high and, thus, constitutes the major part of manager maintenance costs.

[8] Most empiric studies focus on executive's compensation only and do not cover compensation of other managers in a firm. A notable exception is the paper [26] where salaries and other compensations of all managers of General Motors Company during a certain time period are studied.

[9] The results extend immediately to the search of the optimal r-tree (i.e., the tree where each manager has no more than r immediate subordinates).

in each of its arguments, and deleting a manager controlling a group of measure zero does not increase the cost of his immediate superior, there exists an optimal tree (see [20] for details).

All immediate subordinates of a manager in a tree-shaped hierarchy control non-overlapping groups of workers. Thus, the measure μ of the group controlled by the manager is always equal to the sum $\mu_1 + \ldots + \mu_r$ of measures of groups controlled by her immediate subordinates. For simplicity we skip below the argument μ of a measure-dependent cost function assuming the manager cost $c(\mu_1, \ldots, \mu_r)$ to depend only on the measures $\mu_1, \ldots, \mu_r$ of groups controlled by her immediate subordinates.

*Definition 7. An **r-dimensional simplex** D_r is a collection of non-negative r-dimensional vectors $x = (x_1, \ldots, x_r)$ with components summing up to unity*: $x_1 + \ldots + x_r = 1$. *An element of a simplex will be referred to as a **proportion**.*

For several special cases of the considered problem one can suggest an efficient dynamic programming algorithm, which builds the optimal hierarchy [7]. From algorithm runs for different cost functions some general conclusions were made about the shape of the optimal tree. Firstly, every manager in an optimal tree seems to have approximately equal number of immediate subordinates. Secondly, if we take groups of workers controlled by immediate subordinates of any manager *m* in an optimal tree, we see the measures of these groups to follow similar proportion, irrespective of the manager *m*.[10] In the next section we formalize these findings and prove formally the properties of the optimal tree.

5. Results

Introduce the concept of a uniform tree, which plays the central role in the rest of the paper.

*Definition 8. A tree is said to be **(r, x)-uniform** if each of its managers has exactly r immediate subordinates and splits the measure of a subordinate group of workers among them in the proportion* $x = (x_1, \ldots, x_r)$. *The number r is called the **span of control** of the uniform tree.*

Example 2. Fig. 3 shows three uniform trees; numbers in circles represent the measure of the group controlled by the corresponding employee. The hierarchy (a) is a 3-tree with proportion $x = (1/3, 1/3, 1/3)$. The tree is symmetric (actually, uniform trees are always symmetric when workers have equal measures). The hierarchy (b) makes a symmetric 2-tree with proportion (1/2, 1/2), while (c) is an asymmetric 2-tree with proportion (1/3, 2/3).

Since the optimal tree problem is discrete, sometimes no uniform tree exists for the given set of workers (the only exception is a fan, which is always uniform). At the same time, if a uniform tree exists, its cost is easily evaluated.

[10] Under some cost functions the proportion is *symmetric*, i.e., measures of subordinate groups are equalized, but for other cost functions *asymmetric* hierarchies appear to be optimal.

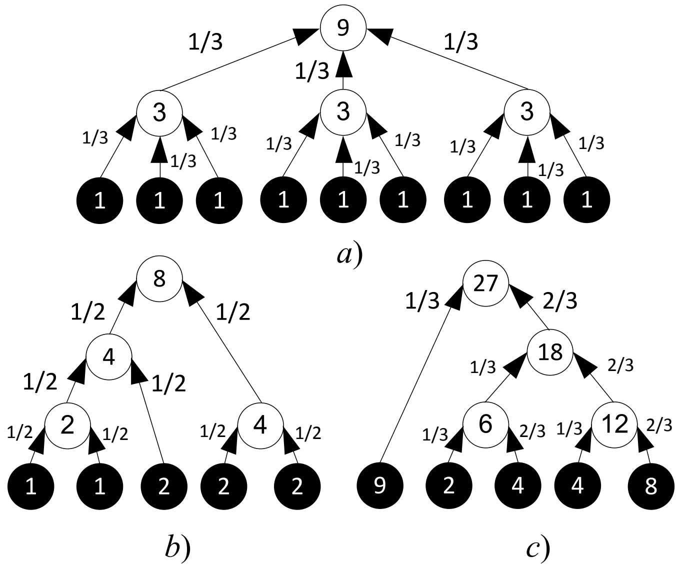

Fig. 3. Examples of uniform trees

Assertion 1. Consider a set of workers $N = \{1, \ldots, n\}$ *with measures* $\mu(1), \ldots, \mu(n)$, *and a homogeneous cost function* $c(\mu_1, \ldots, \mu_r)$ *with degree of homogeneity* γ. *If there exists a uniform tree* H *with the span of control* r *and the proportion* $x = (x_1, \ldots, x_r)$, *then its cost can be written as*

$$C(H) = \begin{cases} |\mu^\gamma - \sum_{j=1}^{n} \mu(j)^\gamma| \dfrac{c(x_1, \ldots, x_r)}{|1 - \sum_{i=1}^{r} x_i^\gamma|}, & \text{when } \gamma \neq 1, \\ (\mu \ln \mu - \sum_{j=1}^{n} \mu(j) \ln \mu(j)) \dfrac{c(x_1, \ldots, x_r)}{-\sum_{i=1}^{r} x_i \ln x_i}, & \text{when } \gamma = 1, \end{cases} \tag{1}$$

where $\mu := \mu(N) = \sum_{i=1}^{n} \mu(i)$ *is the total measure of all workers.*

See the Appendix for the proof.

If measures of all n workers are equal to unity, a simpler expression can be written for the cost of the (r, x)-uniform tree:

$$C(H) = \begin{cases} |n^\gamma - n| \dfrac{c(x_1, \ldots, x_r)}{|1 - \sum_{i=1}^{r} x_i^\gamma|}, & \text{when } \gamma \neq 1, \\ n \ln n \dfrac{c(x_1, \ldots, x_r)}{-\sum_{i=1}^{r} x_i \ln x_i}, & \text{when } \gamma = 1. \end{cases}$$

A uniform tree rarely exists for the given span of control r and the proportion x. Nevertheless, expression (1) can be calculated irrespective of the existence of the corresponding (r, x)-uniform tree, and for any span of control r and any proportion $x \in D_r$ we can evaluate the cost of the (r, x)-uniform tree as if it exists. We can also look for the best parameters of a uniform tree in (1) without any regard to the existence of a tree with such parameters. To find the *best uniform tree* we minimize the expression (1) over all possible spans of control r and proportions x. A tuple (r, x), for which the minimum is attained, gives the parameters of the best uniform tree, and substitution of (r, x) into the expression (1) gives the cost of this tree.

It is clear that a top-manager in any tree over a set of n workers has no more than n immediate subordinates, and, therefore, when looking for the best uniform tree, we can minimize over all r from 2 to n. Moreover, at least one worker is subordinated to every immediate subordinate of the top manager, and, therefore, the measure of the group of workers controlled by this immediate subordinate is not less than the least of workers' measures. Consequently, if workers have measures $\mu(1), \ldots, \mu(n)$, then each component x_i ($i = 1, \ldots, r$) of the proportion in any uniform tree over this set of workers will be at least

$$\varepsilon := \frac{\min_{i \in N} \mu(i)}{\sum_{i \in N} \mu(i)}.$$

For an arbitrary non-negative number ε denote with $D_r(\varepsilon)$ the part of the simplex D_r where each component of the proportion $x \in D_r$ is not less than ε. Then the cost of the best uniform tree over the set of workers N is written as:

$$(2) \qquad C_L(N) := \begin{cases} |\mu^\gamma - \sum_{j=1}^{n} \mu(j)^\gamma| \min_{k=2,\ldots,n} \min_{y \in D_k(\varepsilon)} \dfrac{c(y_1, \ldots, y_k)}{|1 - \sum_{i=1}^{k} y_i^\gamma|}, & \text{when } \gamma \neq 1, \\ (\mu \ln \mu - \sum_{j=1}^{n} \mu(j) \ln \mu(j)) \min_{k=2,\ldots,n} \min_{y \in D_k(\varepsilon)} \dfrac{c(y_1, \ldots, y_k)}{-\sum_{i=1}^{k} y_i \ln y_i}, & \text{when } \gamma = 1, \end{cases}$$

where $\mu = \sum_{i \in N} \mu(i)$, $\varepsilon = \min_{i \in N} \mu(i) / \mu$.

If a set of workers and a manager cost function are given, calculation of the span of control r and the proportion x of the best uniform tree reduces to solution of n minimization problems for a non-linear function over a convex compact set[11].

As the following assertion claims, the cost of the best uniform tree appears to be closely related to the cost of the optimal tree-like hierarchy.

Assertion 2. Let $N = \{1, \ldots, n\}$ be a set of workers with measures $\mu(1), \ldots, \mu(n)$, and consider a homogenous manager cost function $c(\mu_1, \ldots, \mu_r)$ with degree of homogeneity γ. Then the cost of any tree-like hierarchy over the set of workers N is not less than $C_L(N)$, i.e., the cost of the best uniform tree $C_L(N)$ is a lower-bound estimate of the tree cost.

See the Appendix for the proof.

The same argument can be applied to the cost of an r-tree. It is bounded from below with the cost $C_L^r(N)$ of the best uniform r-tree, i.e., the uniform tree with the span of control not exceeding r.

Corollary 1. If the best uniform tree (or r-tree) exists, it is optimal on the set of all trees (or, correspondingly, r-trees).

See the Appendix for the proof.

Therefore, we see that the span of control and the proportion of the optimal tree strive to the corresponding parameters of the best uniform tree, and it is just discreteness of the set of workers that limits this tendency[12].

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[11] Since $D_r(\varepsilon)$ is compact, minima in the expression (2) are achieved under considerably weak restrictions on the cost function (according to [31] the function is enough to be lower semicontinuous) and below we assume the minima are always achieved.

The obtained lower bound of the tree cost has wide range of applications. It is shown in [20] that in many practical situations its value is very close to the cost of the optimal tree when the number of workers is big enough. It is also shown in [20] how this lower-bound estimate is applied in efficient algorithms building nearly optimal trees whose cost slightly exceeds the cost of the optimal tree. In practical calculations we can employ this theory and use the closed-form expression (2) for the lower-bound estimate instead of the exact value of the optimal tree cost.

Also a technique is suggested in [20] to the lower bound calculation and a taxonomy of solutions is suggested. Among others, the following useful lemma holds.

Lemma 1. Assume for each span of control a homogeneous manager cost function achieves its minimum on the simplex when measures of subordinate groups equalize. Then the best uniform tree is symmetric, i.e. all components of the proportion of this tree are equal to each other[13].

See the Appendix for the proof.

6. Example of optimal tree search

In the present section we find the best uniform tree for the multiplicative cost function. The lower bound of the tree cost is calculated and the direct relation is established between the shape of the optimal hierarchy and the parameters of a cost function.[14]

Consider a multiplicative homogeneous cost function $c(r, \mu) = \mu^\gamma \varphi(r)$, where μ is a measure of the group of workers subordinated to the manager, and r is the span of control of this manager. Multiplicative functions play important role in the study of measure-dependent cost functions. From the formula (2) we know that the best uniform tree tends to symmetry (the internal minimum in the expression (2) is likely to be achieved in the center of the simplex). On the other hand, if we consider only symmetric trees, every cost function $c(\mu_1, \ldots, \mu_r)$ with degree of homogeneity γ reduces to the multiplicative function $\mu^\gamma \varphi(r)$, where $\mu := \mu_1 + \ldots + \mu_r$ and $\varphi(r) := c(1/r, \ldots, 1/r)$. Thus, construction of the best symmetric uniform tree reduces to the analysis of some multiplicative cost function.

The function $\mu^\gamma \varphi(r)$ does not depend on the proportion, and, thus, it is equal to $\varphi(r)$ on the simplex D_r. Consequently, this (constant) function is convex on the simplex and, according to Lemma Lemma *1*, the best uniform tree is symmetric, i.e., each manager divides the measure of her subordinate group equally among her immediate subordinates. So, we are left to find the best span of control to determine all parameters of the best uniform tree. According to the expression (2), $\gamma \neq 1$, we need to find the number $r = 2, \ldots, n$, which minimizes the following expression:

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[12] A prototype of AssertionAssertion 2 has been proved in [**Ошибка! Источник ссылки не найден.**] for the case of a continuous set of workers. The optimal tree has been shown to be uniform in this case. Nevertheless, results of [**Ошибка! Источник ссылки не найден.**] are limited to cost functions with degree of homogeneity exceeding unity.

[13] In particular, Lemma holds for the manager cost function being convex on the simplex D_r for any span of control r.

[14] As shown in [20], only a tree-shaped hierarchy can be optimal for the multiplicative manager cost function.

(3) $$\frac{\varphi(r)}{|1-r^{1-\gamma}|}.$$

The case of $\gamma = 1$ is studied by analogy.

Multiplicative functions with $\varphi(r) = r^\beta$ arise naturally in many applications studied in [8, 9, 20]. In this case from the first-order conditions we find the span of control of the best uniform tree to be one of two integers nearest to

$$\left(\frac{\beta}{\beta+\gamma-1}\right)^{\frac{1}{1-\gamma}}.$$

In Fig. 4 we show how the optimal span of control depends on the values of parameters β and γ of the manager cost function. 2-trees appear to be optimal for large β and γ (the area where they are optimal is labeled with the number "2" in the figure). As soon as β and γ decrease, 3-trees, 4-trees, etc, become optimal (these areas are labeled with "3", "4", etc, respectively). When $\beta + \gamma < 1$ it is shown in [20] that the optimal hierarchy is a fan where the sole manager controls all workers immediately (this area is labeled with the sign "∞" in the figure).

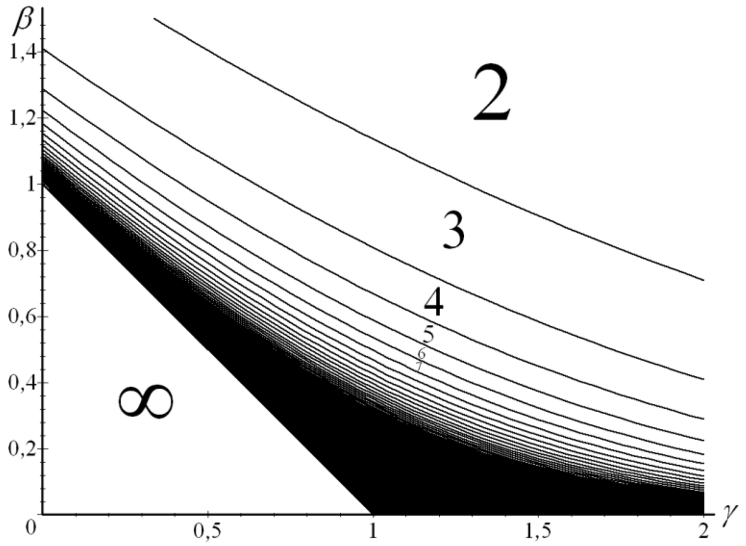

Fig. 4. Optimal span of control for different combinations of cost function parameters

Boundaries of areas where different spans of control are optimal are easily calculated. From the expression (3) it follows that the equality for the border of areas where r-trees and $(r + 1)$-trees are optimal can be written as

$$r^\beta \big/ |1-r^{1-\gamma}| = (r+1)^\beta \big/ |1-(r+1)^{1-\gamma}|.$$

Resolving this equation for β we obtain

$$\beta = \ln\left|\frac{1-(r+1)^{1-\gamma}}{1-r^{1-\gamma}}\right| \bigg/ \ln\left(\frac{r+1}{r}\right).$$

Substituting $r = 2$ in this formula obtain the equation of the border between the areas where 2-trees and 3-trees are optimal. Substitution of $r = 3$ gives the border between the areas where 3-trees and 4-trees are optimal, etc.

So, the best uniform tree appears symmetric, and its span of control for fixed β and γ can be found from Fig. 4. Substituting the optimal span of control and the symmetric proportion into the expression (1) obtain a lower-bound estimate of the tree cost under a multiplicative manager cost function. Efficient algorithms from [20] can be used to construct nearly optimal hierarchies. These hierarchies will be approximately symmetric, and spans of control of its managers will be approximately equal to the span of control of the best uniform tree.

7. Conclusion

In the present paper we set and solve the problem of the optimal tree-shaped hierarchy for the case of homogeneous manager cost function. We show that all managers in an optimal tree have approximately equal spans of control and divide the measure of the subordinate group among their immediate subordinates in roughly the same proportion. The main result of the paper is the closed-form expression for the lower-bound estimate of the cost of a tree-shaped hierarchy. Combined with the criteria of the good quality of this estimate and with efficient routines from [20], which build nearly optimal trees, these results solve the optimal tree problem for a homogeneous cost function, at least, from the practical point of view.

Among the future directions we would mention the further development of the theory and parameter identification of manager cost functions from datasets available in industry. If real-life manager cost functions fall into the class of functions where the theory can be applied, the results of this paper can be applied to organizational structure design in industry, public administration, state and local government.

Appendix

In the proofs below the following inequalities are used being a special case of Minkovsky inequality [33]: for arbitrary non-negative numbers $x_1, \ldots, x_k$ and for any number γ

(A.1) $$(x_1 + \ldots + x_k)^\gamma \geq x_1^\gamma + \ldots + x_k^\gamma, \text{ when } \gamma \geq 1,$$

(A.2) $$(x_1 + \ldots + x_k)^\gamma \leq x_1^\gamma + \ldots + x_k^\gamma, \text{ when } \gamma \leq 1.$$

Proof of Assertion The tree is symmetric (actually, uniform trees are always symmetric when workers have equal measures). The hierarchy (b) makes a symmetric 2-tree with proportion (1/2, 1/2), while (c) is an asymmetric 2-tree with proportion (1/3, 2/3).

Since the optimal tree problem is discrete, sometimes no uniform tree exists for the given set of workers (the only exception is a fan, which is always uniform). At the same time, if a uniform tree exists, its cost is easily evaluated.. The proof employs induction on the number of workers n. For a single worker (i.e., when $n = 1$) only a tree without managers is possible. Below we treat this "tree" as a uniform one. The cost of this "tree" is equal to zero. The expression (1) is also equal to zero, therefore, it holds for $n = 1$. Assume the assertion is proved for any number of workers less than n. Let us show it holds also for the set of workers consisting of n elements.

A top-manager in an (r, x)-uniform tree H controls a group consisting of all workers. Let it have the measure μ. Immediate subordinates of the top-manager control groups $s_1, \ldots, s_r$ with measures $\mu_k = x_k\mu$, $k = 1, \ldots, r$. The cost $C(H)$ of the tree H adds up from the cost of the top-manager and the costs of r sub-trees $H_1, \ldots, H_r$, rooted by her immediate subordinates. Since each sub-tree H_k is also a uniform tree for the set of workers s_k, $k = 1, \ldots, r$, and each group s_k contains less than n workers, by inductive assumption we can write the cost of the tree H as

$$C(H) = c(\mu_1, \ldots, \mu_r) + C(H_1) + \ldots + C(H_r),$$

where $C(H_k)$, $k = 1, \ldots, r$, is defined with the expression (1).

Introduce a shorthand notation $C := c(x_1, \ldots, x_r)$. Since the cost function is homogeneous, we write $c(\mu_1, \ldots, \mu_r) = \mu^\gamma c(x_1, \ldots, x_r) = \mu^\gamma C$. Consider the case of $\gamma \neq 1$ first. We have

$$C(H) = \mu^\gamma C + \sum_{k=1}^{r} |\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma| \frac{C}{|1 - \sum_{i=1}^{r} x_i^\gamma|} = C \frac{\mu^\gamma |1 - \sum_{i=1}^{r} x_i^\gamma| + \sum_{k=1}^{r} |\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma|}{|1 - \sum_{i=1}^{r} x_i^\gamma|} =$$

$$= C \frac{|\mu^\gamma - \sum_{i=1}^{r} \mu^\gamma x_i^\gamma| + \sum_{k=1}^{r} |\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma|}{|1 - \sum_{i=1}^{r} x_i^\gamma|} = C \frac{|\mu^\gamma - \sum_{i=1}^{r} \mu_i{}^\gamma| + \sum_{k=1}^{r} |\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma|}{|1 - \sum_{i=1}^{r} x_i^\gamma|}.$$

From (A.1) and (A.2) it follows that expressions $\mu^\gamma - \sum_{i=1}^{r} \mu_i{}^\gamma$ and $\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma$ are either both positive or are both negative, and, therefore,

$$|\mu^\gamma - \sum_{i=1}^{r} \mu_i{}^\gamma| + \sum_{k=1}^{r} |\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma| = |\mu^\gamma - \sum_{i=1}^{r} \mu_i{}^\gamma + \sum_{k=1}^{r} (\mu_k{}^\gamma - \sum_{j \in s_k} \mu(j)^\gamma)| = |\mu^\gamma - \sum_{j=1}^{n} \mu(j)^\gamma|.$$

Consequently,

$$C(H) = C \frac{|\mu^\gamma - \sum_{j=1}^{n} \mu(j)^\gamma|}{|1 - \sum_{i=1}^{r} x_i^\gamma|} = |\mu^\gamma - \sum_{j=1}^{n} \mu(j)^\gamma| \frac{c(x_1, \ldots, x_r)}{|1 - \sum_{i=1}^{r} x_i^\gamma|},$$

and, therefore, the expression (1) holds if $\gamma \neq 1$.

Now consider $\gamma = 1$. Then, in the same fashion,

$$C(H) = \mu C + \sum_{k=1}^{r} (\mu_k \ln \mu_k - \sum_{j \in s_k} \mu(j) \ln \mu(j)) \frac{C}{-\sum_{i=1}^{r} x_i \ln x_i} =$$

$$= C \frac{-\mu \sum_{i=1}^{r} x_i \ln x_i + \sum_{k=1}^{r} (\mu_k \ln \mu_k - \sum_{j \in s_k} \mu(j) \ln \mu(j))}{-\sum_{i=1}^{r} x_i \ln x_i} = C \frac{\sum_{i=1}^{r} [\mu_i \ln \mu_i - \mu_i \ln x_i - \sum_{j \in s_i} \mu(j) \ln \mu(j)]}{-\sum_{i=1}^{r} x_i \ln x_i} =$$

$$= C \frac{\sum_{i=1}^{r} [\mu_i \ln(\mu_i / x_i) - \sum_{j \in s_i} \mu(j) \ln \mu(j)]}{-\sum_{i=1}^{r} x_i \ln x_i} = C \frac{\sum_{i=1}^{r} \mu_i \ln(\mu) - \sum_{j=1}^{n} \mu(j) \ln \mu(j)}{-\sum_{i=1}^{r} x_i \ln x_i} =$$

$$= C\frac{\ln(\mu)\sum_{i=1}^{r}\mu_i - \sum_{j=1}^{n}\mu(j)\ln\mu(j)}{-\sum_{i=1}^{r}x_i\ln x_i} = (\mu\ln\mu - \sum_{j=1}^{n}\mu(j)\ln\mu(j))\frac{c(x_1,...,x_r)}{-\sum_{i=1}^{r}x_i\ln x_i}.$$

Therefore, we proved Assertion The tree is symmetric (actually, uniform trees are always symmetric when workers have equal measures). The hierarchy (b) makes a symmetric 2-tree with proportion (1/2, 1/2), while (c) is an asymmetric 2-tree with proportion (1/3, 2/3).

Since the optimal tree problem is discrete, sometimes no uniform tree exists for the given set of workers (the only exception is a fan, which is always uniform). At the same time, if a uniform tree exists, its cost is easily evaluated..

Proof of AssertionAssertion 2. We follow the line of Assertion The tree is symmetric (actually, uniform trees are always symmetric when workers have equal measures). The hierarchy (b) makes a symmetric 2-tree with proportion (1/2, 1/2), while (c) is an asymmetric 2-tree with proportion (1/3, 2/3).

Since the optimal tree problem is discrete, sometimes no uniform tree exists for the given set of workers (the only exception is a fan, which is always uniform). At the same time, if a uniform tree exists, its cost is easily evaluated. proof and employ induction on the number of workers n. Consider a single worker having the measure μ_1. It is clear from (2) that $C_L(\mu_1) = 0$, which is equal to the cost of the sole admissible "tree" over a single worker (it contains just this worker and no managers). Assume the Assertion holds for any number of workers less than n and show that it also holds for the set of workers N consisting of n workers.

Consider an arbitrary tree-shaped hierarchy H over the set of workers N and let its top-manager have k immediate subordinates controlling groups of workers $s_1, \dots, s_k$ of measures $\mu_1, \dots, \mu_k$ respectively. The cost of the tree H adds up from the cost of the top-manager and the costs of the sub-trees $H_1, \dots, H_k$, rooted by her immediate subordinates (the cost of a sub-tree consisting of a single worker is equal to zero):

$$C(H) = c(\mu_1,...,\mu_k) + C(H_1) + ... + C(H_k).$$

Since any group s_i, $i = 1, \dots, k$, contains less than n workers, by inductive assumption the cost of the corresponding sub-tree cannot be less than $C_L(s_i)$. Consequently,

$$C(H) \geq c(\mu_1,...,\mu_k) + C_L(s_1) + ... + C_L(s_k).$$

In the right-hand side of this inequality we see the fixed number of immediate subordinates k and the fixed partition $s_1, \dots, s_k$ of the set of workers N into k subsets. Consequently, the right-hand side will not increase if we take the minimum over all k from 2 to n and over all possible partitions $s_1, \dots, s_k$ of the set N of workers into k pieces. So, we obtain

$$C(H) \geq \min_{k=2,\dots,n} \min_{\substack{s_1,\dots,s_k: \\ \bigcup_{i=1}^{k} s_i = N}} \{c(s_1,...,s_k) + \sum_{i=1}^{k} C_L(s_i)\}.$$

Introduce a shorthand notation

(A.3) $$F(n,\varepsilon) := \begin{cases} \min\limits_{k=2,\ldots,n} \min\limits_{y \in D_k(\varepsilon)} \dfrac{c(y_1,\ldots,y_k)}{|1-\sum_{i=1}^k y_i^\gamma|}, & \text{when } \gamma \neq 1, \\ \min\limits_{k=2,\ldots,n} \min\limits_{y \in D_k(\varepsilon)} \dfrac{c(y_1,\ldots,y_k)}{-\sum_{i=1}^k y_i \ln y_i}, & \text{when } \gamma = 1. \end{cases}$$

It can be used to write the expression (2) in compact form as

(A.4) $$C_L(N) = \begin{cases} |\mu^\gamma - \sum_{j=1}^n \mu(j)^\gamma| F(n,\varepsilon), & \text{when } \gamma \neq 1, \\ (\mu \ln \mu - \sum_{j=1}^n \mu(j) \ln \mu(j)) F(n,\varepsilon), & \text{when } \gamma = 1. \end{cases}$$

Let $r(n, \varepsilon)$ be the span of control and let $x(n, \varepsilon) = (x_1(n, \varepsilon), \ldots, x_{r(n, \varepsilon)}(n, \varepsilon))$ be the proportion where the minimum in (A.3) is achieved.

Assume the degree of homogeneity $\gamma \neq 1$. Then it follows from (A.4) that

(A.5) $$C(H) \geq \min_{k=2,\ldots,n} \min_{\substack{s_1,\ldots,s_k: \\ \bigcup_{i=1}^k s_i = N}} \{c(\mu_1,\ldots,\mu_k) + \sum_{i=1}^k |\mu_i^\gamma - \sum_{j \in s_i} \mu(j)^\gamma| F(n_i,\varepsilon_i)\},$$

where $\mu_1, \ldots, \mu_k$ are the measures of the groups $s_1, \ldots, s_k$ respectively, and $n_1, \ldots, n_k$ are cardinalities of these groups, while $\varepsilon_i = \min\limits_{j \in s_i} \mu(j)/\mu_i$.

It is clear that the function $F(n, \varepsilon)$ does not increase in the number of workers n, since the minimization domain in (A.3) extends when n increases. For the same reason the function $F(n, \varepsilon)$ does not decrease in ε.

Since $n_i < n$ and $\varepsilon_i \geq \varepsilon = \min_{j \in N} \mu(j)/\sum_{j \in N} \mu(j)$, the right-hand side in the inequality (A.5) can only decrease if we replace n_i with n and replace ε_i with ε. Therefore, we obtain

$$C(H) \geq \min_{k=2,\ldots,n} \min_{\substack{s_1,\ldots,s_k: \\ \bigcup_{i=1}^k s_i = N}} \{c(\mu_1,\ldots,\mu_k) + F(n,\varepsilon) \sum_{i=1}^k |\mu_i^\gamma - \sum_{j \in s_i} \mu(j)^\gamma|\} =$$

$$= \min_{k=2,\ldots,n} \min_{\substack{s_1,\ldots,s_k: \\ \bigcup_{i=1}^k s_i = N}} \{c(\mu_1,\ldots,\mu_k) + F(n,\varepsilon) |\sum_{i=1}^k \mu_i^\gamma - \sum_{j \in N} \mu(j)^\gamma|\}.$$

Now the right-hand side depends not on the groups $s_1, \ldots, s_k$ directly but rather on their measures $\mu_1, \ldots, \mu_k$. Therefore, the right-hand side can only further decrease if we replace the minimum over all partitions $s_1, \ldots, s_k$ with the minimum over all measures $\mu_1, \ldots, \mu_k$, such that $\mu_i \geq \mu\varepsilon$, $i = 1, \ldots, k$, and the sum $\mu_1 + \ldots + \mu_k$ is equal to $\mu := \mu(N)$. Then we have

$$C(H) \geq \min_{k=2,\ldots,n} \min_{\substack{\mu_i \geq \mu \cdot \varepsilon: \\ \sum_{i=1}^k \mu_i = \mu}} \{c(\mu_1,\ldots,\mu_k) + F(n,\varepsilon) |\sum_{i=1}^k \mu_i^\gamma - \sum_{j \in N} \mu(j)^\gamma|\} =$$

$$= \mu^\gamma \min_{k=2,\ldots,n} \min_{y \in D_k(\varepsilon)} \{c(y_1,\ldots,y_k) + F(n,\varepsilon) |\sum_{i=1}^k y_i^\gamma - \sum_{j \in N} (\mu(j)/\mu)^\gamma|\}.$$

Add and subtract $C_L(N)$ in the right-hand side of this inequality:

$$C(H) \geq C_L(N) + \min_{k=2,\ldots,n} \min_{y \in D_k(\varepsilon)} \{\mu^\gamma c(y_1,\ldots,y_k) + F(n,\varepsilon)[|\sum_{i=1}^k \mu^\gamma y_i^\gamma - \sum_{j \in N} \mu(j)^\gamma| - |\mu^\gamma - \sum_{j \in N} \mu(j)^\gamma|]\}.$$

Note that from inequalities (A.1) and (A.2) it follows that the expressions $\sum_{i=1}^{k}\mu^\gamma y_i^\gamma - \sum_{j\in N}\mu(j)^\gamma$ and $\mu^\gamma - \sum_{j=1}^{n}\mu(j)^\gamma$ are either both positive or are both negative. Moreover, for $\gamma > 1$ we have $\sum_{i=1}^{k}\mu^\gamma y_i^\gamma \le \mu^\gamma$, and for $\gamma < 1$ we have $\sum_{i=1}^{k}\mu^\gamma y_i^\gamma \ge \mu^\gamma$. Consequently, irrespective of the value of γ the following identities hold,

$$|\sum_{i=1}^{k}\mu^\gamma y_i^\gamma - \sum_{j\in N}\mu(j)^\gamma| - |\mu^\gamma - \sum_{j\in N}\mu(j)^\gamma| \equiv -|\mu^\gamma - \sum_{i=1}^{k}\mu^\gamma y_i^\gamma| \equiv -\mu^\gamma\,|1-\sum_{i=1}^{k}y_i^\gamma|,$$

and the above inequality can be rewritten as

$$C(H) \ge C_L(N) + \min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\{\mu^\gamma c(y_1,\dots,y_k) - F(n,\varepsilon)\mu^\gamma\,|1-\sum_{i=1}^{k}y_i^\gamma|\},$$

or, equivalently,

$$C(H) \ge C_L(N) + \mu^\gamma\min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}|1-\sum_{i=1}^{k}y_i^\gamma|\,\{\frac{c(y_1,\dots,y_k)}{|1-\sum_{i=1}^{k}y_i^\gamma|} - F(n,\varepsilon)\}.$$

The first multiplier in the minimized function is obviously non-negative. The second multiplier is also non-negative, since, according to (A.3), it achieves its minimum (being equal to zero) under $k = r(n, \varepsilon)$ and $y = x(n, \varepsilon)$. Therefore, the minimum in the right-hand side of the inequality is equal to zero. Hence, $C(H) \ge C_L(N)$, and the assertion holds for $\gamma \ne 1$.

Now consider $\gamma = 1$. In this case the expression (A.5) is written as

$$C(H) \ge \min_{k=2,\dots,n}\min_{\substack{s_1,\dots,s_k:\\ \bigcup_{i=1}^{k}s_i=N}}\{c(\mu_1,\dots,\mu_k) + \sum_{i=1}^{k}[\mu_i\ln\mu_i - \sum_{j\in s_i}\mu(j)\ln\mu(j)]F(n_i,\varepsilon_i)\}.$$

In the same manner we relax this inequality by replacing n_i with n and ε_i – with ε, which extends the minimization domain. Then we obtain the following inequality:

$$C(H) \ge \min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\{\mu c(y_1,\dots,y_k) + F(n,\varepsilon)(\sum_{i=1}^{k}\mu y_i\ln\mu y_i - \sum_{j\in N}\mu(j)\ln\mu(j))\}.$$

Add and subtract $C_L(N)$ in the right-hand side:

$$C(H) \ge C_L(N) + \min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\{\mu c(y_1,\dots,y_k) + F(n,\varepsilon)(\sum_{i=1}^{k}\mu y_i\ln\mu y_i -$$
$$-\sum_{j\in N}\mu(j)\ln\mu(j) - \mu\ln\mu + \sum_{j\in N}\mu(j)\ln\mu(j))\} =$$
$$= C_L(N) + \min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\{\mu c(y_1,\dots,y_k) + F(n,\varepsilon)(\sum_{i=1}^{k}\mu y_i\ln\mu y_i - \mu\ln\mu)\} =$$
$$= C_L(N) + \mu\min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\{c(y_1,\dots,y_k) + F(n,\varepsilon)\sum_{i=1}^{k}y_i\ln y_i\}.$$

Take the non-negative term $\left(-\sum_{i=1}^{k}y_i\ln y_i\right)$ out of brackets:

$$C(H) \ge C_L(N) + \mu\min_{k=2,\dots,n}\min_{y\in D_k(\varepsilon)}\left(-\sum_{i=1}^{k}y_i\ln y_i\right)\left\{\frac{c(y_1,\dots,y_k)}{-\sum_{i=1}^{k}y_i\ln y_i} - F(n,\varepsilon)\right\}.$$

Hence, the first multiplier in the minimized function is non-negative, and so is the second multiplier, since it achieves its minimum (being equal to zero) under $k = r(n, \varepsilon)$, $y = x(n, \varepsilon)$. Therefore, the minimum in the right-hand side is equal to zero, and $C(H) \geq C_L(N)$ for $\gamma = 1$.

This finishes the proof of AssertionAssertion *2*.

*Proof of Corollary Corollary 1***.** It follows from AssertionAssertion *2* that the cost of any tree (consequently, any r-tree) is not less than $C_L(N)$ (consequently, $C_L^r(N)$). But if the best uniform tree (r-tree) exists, then, by Assertion The tree is symmetric (actually, uniform trees are always symmetric when workers have equal measures). The hierarchy (b) makes a symmetric 2-tree with proportion (1/2, 1/2), while (c) is an asymmetric 2-tree with proportion (1/3, 2/3).

Since the optimal tree problem is discrete, sometimes no uniform tree exists for the given set of workers (the only exception is a fan, which is always uniform). At the same time, if a uniform tree exists, its cost is easily evaluated., its cost is equal to $C_L(N)$ (consequently, to $C_L^r(N)$), and, hence, this tree is optimal.

Proof of Lemma Lemma 1. For an arbitrary span of control r the denominator of the minimized function in the expression (2) achieves its maximum in the center of the simplex, at the point $y = (1/r, \ldots, 1/r)$ where all vector components equalize. Hence, if the minimum of the numerator $c(y_1, \ldots, y_k)$ is achieved at the same point, then such a symmetric proportion y delivers the minimum to the whole ratio. This proves the lemma.

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