In Game-Theoretic Models in Mathematical Ecology / Editors: V. Mazalov, D. Novikov, G. Ougolnitsky, L. Petrosyan. – New York: Nova Science Publishers, 2015. – P. 1 – 16. ISBN 978-1-63483-489-6.

MODELS OF COLLECTIVE THRESHOLD BEHAVIOR IN CONTROL PROBLEMS OF ECOLOGICAL-ECONOMIC SYSTEMS

V.V. Breer, D.A. Novikov, and A.D. Rogatkin
(Trapeznikov Institute of Control Sciences,
Russian Academy of Sciences, Moscow, Russia)

We study models of collective "multi-threshold" behavior of agents making binary decisions. The general analysis scheme of these models is applied to three control problems of ecological-economic systems, namely, the problem of individual penalties, the problem of individual and collective penalties, as well as the incentive problem of agents' investments in environmental protection and/or restoration.

Keywords: threshold behavior, discrete dynamical system, control of ecological-economic systems.

1. Introduction

Since the classical works of M. Granovetter [15] and T. Schelling [22], much attention of researchers in the field of collective behavior models focuses on the following settings. While making binary decisions on their "activity" or "passivity," agents observe the opponents' action profile, i.e., the number of active or passive opponents/"neighbors" (see the survey [2]). In this case, a key characteristic of an agent is its threshold: the agent varies its behavior as the opponents' action profile exceeds the threshold. Such models provide an adequate description to the effects of conformity and anticonformity behavior (both in terms of collective behavior theory and game theory [11]) and have many applications (social networks [1], mob control [12] and others [2]).

The general modeling procedure is as follows. First, construct the goal functions of all agents using the practical interpretations of socioeconomic phenomena and processes in question. Second, find the best responses of the agents (the relationships between their actions maximizing the goal functions and the opponents' actions). And third, perform transition to the discrete dynamical system describing the evolution of the number or share of active agents (the right-hand side of this dynamical system is defined by the agents' threshold distribution function). We refer to [2] for numerous examples.

However, some real situations are hardly reflected by the simple model with a single threshold predetermining agent's activity or passivity under a given opponents' action profile. This leads to the need for extending the class of threshold behavior models owing to the assumption of multiple thresholds for each agent. For instance, an agent can demonstrate conformity under a small number of active opponents, whereas a large number of active opponents can cause its anticonformity. Moreover, it is desired to cover a wider class of situations when an agent decides to be active provided that the share of its active opponents belongs to a given set.

Section 2 states the general design scheme of such "multi-threshold" models. The obtained results are then applied to formulate and study control problems of ecological-economic systems (Sections 3-5), where agents make decisions under penalties (incentives) for environmental protection and/or restoration.

As a matter of fact, this paper extends the models of threshold behavior to the case of n thresholds considered by an agent in its decision-making. On the other hand, the results presented below can be treated as an extension of the optimization [7, 8, 9, 16, 23] and game-theoretic [5, 6, 9, 10, 13, 21] control models of ecological-economic systems, particularly, their control mechanisms [14].

2. Models of collective "multi-threshold" behavior

Consider a set $N = \{1, \ldots, n\}$ of economic agents making binary decisions $y_i \in \{0; 1\}$. If an agent chooses "1," we say that it is "active" (and "passive," otherwise). Introduce the following notation: $Y = \sum_{j \in N} y_j$, $Y_i = \sum_{j \neq i} y_j$, $y = (y_1, \ldots, y_n)$, $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ as the opponents' action profile for agent i, $x = Y / n$ as the share of active agents, and $x_{-i} = Y_{-i} / n$.

Generally, the goal function $f_i(y_i, y_{-i})$ of agent i depends on its own actions and the actions of other agents. Suppose that *caeteris paribus* an agent prefers being active. Direct comparison of the values $f_i(0, y_{-i})$ and $f_i(1, y_{-i})$ yields the best response of agent i to a given opponents' action profile:

(1) $y_i = BR_i(y_{-i}) = \begin{cases} 1, \text{ if } f_i(1, y_{-i}) \geq f_i(0, y_{-i}), \\ 0, \text{ if } f_i(1, y_{-i}) < f_i(0, y_{-i}). \end{cases}$

Further analysis focuses on a class of collective behavior models, where the inequalities in the right-hand side are predetermined only by the number of active opponents of a given agent. Then the best response can be expressed in terms of the share of active agents:

(2) $y_i = \begin{cases} 1, \text{ if } x_{-i} \in A_i, \\ 0, \text{ if } x_{-i} \notin A_i, \end{cases}$

where $A_i \subseteq [0; 1]$ is a subset of the unit segment found from (1).

For instance, in threshold models of conformity behavior [3] we have

(3) $y_i = \begin{cases} 1, \text{ if } x_{-i} \geq \theta_i, \\ 0, \text{ if } x_{-i} < \theta_i, \end{cases}$

where $\theta_i \in [0; 1]$ indicates the so-called conformity threshold of the agent [2, 3, 11, 15]. By virtue of (3), the empirical conformity threshold distribution function $F_n(x) = \frac{1}{n}\left| \{ i \in N : \theta_i < x \} \right|$ can be used to evaluate the share of active agents in a Nash equilibrium $x^* : F_n(x^*) = x^*$ (for details, we refer to [3]). Assume that we know its theoretical counterpart F: [0; 1] → [0; 1] and the initial share $x^0 \in [0; 1]$ of active agents. Then for sufficiently many agents the share of active agents evolves according to the discrete dynamical system

(4) $x^k = F(x^{k-1})$,

where $k = 1, 2, \ldots$ stands for time steps.

In the case of the so-called anticonformity behavior [11],

(5) $y_i = \begin{cases} 1, \text{ if } x_{-i} \leq \varphi_i, \\ 0, \text{ if } x_{-i} > \varphi_i, \end{cases}$

where $\varphi_i \in [0; 1]$ designates the anticonformity threshold of the agent. Similarly to conformity behavior, the expression (5) implies the following. The knowledge of the empirical anticonformity threshold distribution function $G_n(x) = \frac{1}{n}\left| \{ i \in N : \varphi_i < x \} \right|$ allows obtaining the share of active agents in a Nash equilibrium. Again, using its theoretical counterpart G: [0; 1] → [0; 1] and the initial share $x^0 \in [0; 1]$ of active agents, for sufficiently many agents we can calculate

(6) $x^k = 1 - G(x^{k-1})$,

If $A_i = [\theta_i; \varphi_i]$, i.e., agents demonstrate multi-threshold (more specifically, double-threshold) behavior, then

(7) $y_i = \begin{cases} 1, \text{if } x_{-i} \in [\theta_i; \varphi_i], \\ 0, \text{if } x_{-i} \notin [\theta_i; \varphi_i]. \end{cases}$

Obviously, the number of active agents is described by the following discrete dynamical system:

(8) $x^k = F(x^{k-1}) - G(x^{k-1})$.

The sets $\{A_i\}$ may have a more complex structure (e.g., be disconnected– see formula (14) in Section 4). In such situations, the corresponding dynamical system is constructed by analogy.

Any of the discrete dynamical systems (4), (6) or (8), etc. being available, we can perform stability analysis, study how its equilibrium states depend on the model parameters and initial conditions, and so on. Consequently, it is possible to formulate and solve, e.g., parametric control problems (choose admissible values of control parameters ensuring required or almost required dynamics of the system).

In Sections 3-5, we apply the described general design scheme of multi-threshold collective behavior models to three control problems of ecological-economic systems.

3. Model of individual penalties

Within the framework of the control problems of ecological-economic systems studied below, agents' actions will be interpreted as investing (or not investing) fixed financial resources $\{c_i\}$ in environmental protection and/or recovery measures.

Suppose that the goal functions of agents possess the form

(9) $f_i(y) = H_i - c_i\, y_i - \gamma H_i\, \dfrac{1 - y_i}{N - Y}\, I(Y < \hat{Y})$,

where H_i is the income of agent i gained by its business activity, $I(\cdot)$ means the indicator function, γH_i specifies the penalty imposed on the agent for not investing in environmental measures (the total number of such agents does not exceed some threshold $\hat{Y} \le N$). Agents know this threshold for sure or have certain "probabilistic" beliefs about it (see below). The quantities $\gamma \ge 1$ and

$\frac{1}{N-Y}$ can be interpreted as the "penalty strength" and "the audit probability" of an agent.

By evaluating the best response of agent i, we arrive at the following result in terms of the shares x_{-i} of active agents:

$$(10)\ y_i = \begin{cases} 1, \text{ if } x_{-i} \in [1-\gamma\rho_i; \hat{x}], \\ 0, \text{ otherwise.} \end{cases}$$

Here $\hat{x} = \hat{Y} / N \in [0; 1]$ and $\rho_i = H_i / c_i$ (agent's "profitability").

The values $1 - \gamma\rho_i$ and $\hat{x}$ represent the conformity and anticonformity thresholds of the agent.

Let $F(\cdot)$ be the profitability distribution function of the agents and $G(\cdot)$ denote the distribution function of the agents' beliefs about $\hat{x}$. By assumption, both functions form the common knowledge of all agents. According to (8), the share of agents investing their financial resources in environmental measures meets the condition

$$(11)\ x^k = H(x^{k-1}) = \max \{0; 1 - F(\frac{1-x^{k-1}}{\gamma}) - G(x^{k-1})\},$$

where $k = 1, 2, \ldots$ are time steps. The initial share x^0 of such agents is given.

The role of control parameters in this model belongs to "the penalty strength" γ and agents' awareness about the threshold $\hat{x}$.

<u>Example 1.</u> Set $F(z) = \sqrt{z}$, $G(z) = z$, $x^0 = 0.1$. Then the system (11) has the equilibrium 0.25 (see point A in Fig. 1 and Fig. 2).

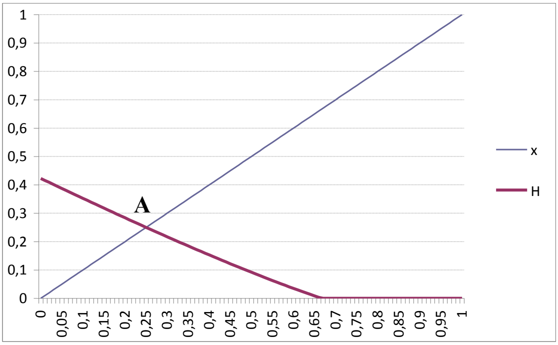

Fig. 1. The right-hand side of the expression (11) in Example 1 under $\gamma = 3$ and $G(z) = z$

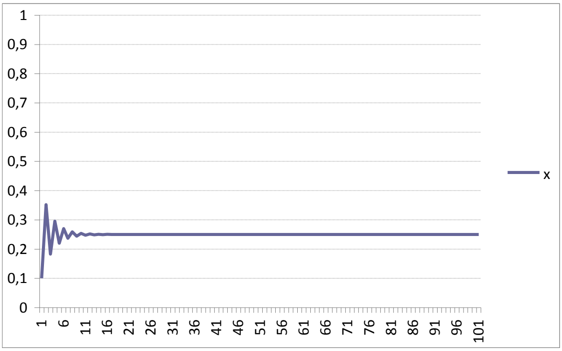

Fig. 2. The trajectory of the system (11) *in Example 1 under* $\gamma = 3$ *and G*(*z*) = *z*

Now, suppose that the agents know the value $\hat{x} = 0.8$ for sure. Under $\gamma = 1$, the system (11) admits the trivial equilibrium. As we increase the penalty strength, the equilibrium also grows. For instance, in the case of $\gamma = 3$ the equilibrium approaches 0.67 (see point A in Fig. 3 and Fig. 4). Interestingly, uncertainty reduction enlarges the share of active agents (we have initially hypothesized that, according to the agents' beliefs, the parameter $\hat{x}$ is uniformly distributed on the unit segment; now, we suppose that the agents know its value for sure).

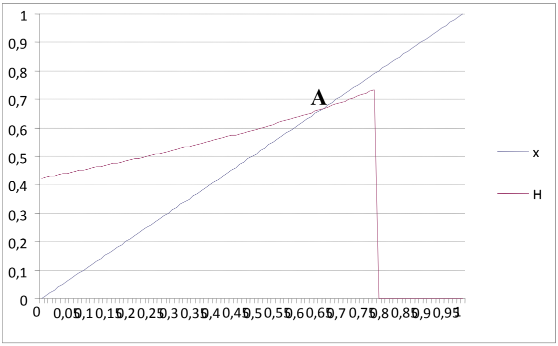

Fig. 3. The right-hand side of the expression (11) *in Example 1 under* $\gamma = 3$ *and* $\hat{x} = 0.8$

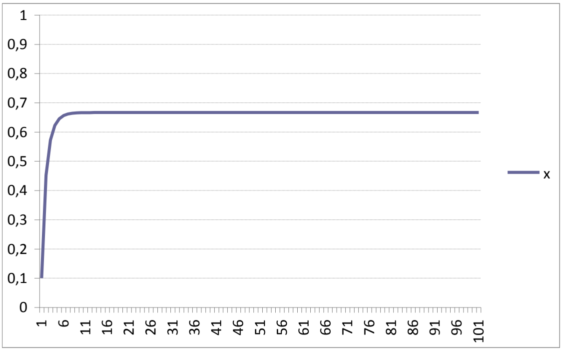

Fig. 4. The trajectory of the system (*11*) *in Example 1*
under $\gamma = 3$ *and* $\hat{x} = 0.8$

Appreciably high "penalty strength" in this model seems unreasonable, as it causes the oscillatory mode (the trajectory of the system (11) under $\gamma = 6$ is illustrated by Fig. 5). •

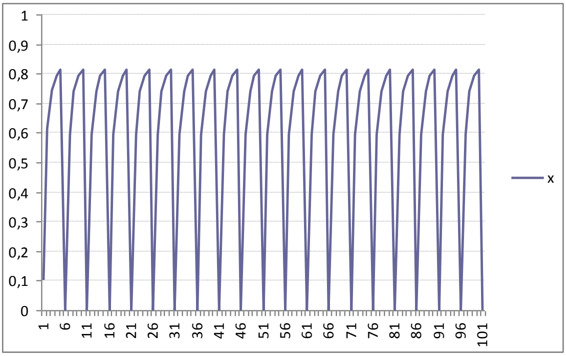

Fig. 5. The trajectory of the system (*11*) *in Example 1*
under $\gamma = 6$ *and* $\hat{x} = 0.8$

Example 2. Assume that agents' profitabilities obey the Pareto distribution with an index α and a minimum possible value ρ_0 (actually, this distribution is widespread in mathematical economics and admits simple identification–see [17, 19]). Agents know the value $\hat{x}$ for sure. Then the expression (11) acquires the form

$$(12)\ x^k = \begin{cases} \left(\dfrac{\gamma\rho_0}{1-x^{k-1}}\right)^{\alpha}, \text{ if } x^{k-1} \le \min\{1-\gamma\rho_0; \hat{x}\}, \\ 0, \text{ otherwise.} \end{cases}$$

Select $\alpha = 2$, $\hat{x} = 0.8$, $\gamma = 1$, and $\rho_0 = 0.3$. The right-hand side of (12) has the curve demonstrated by Fig. 8.

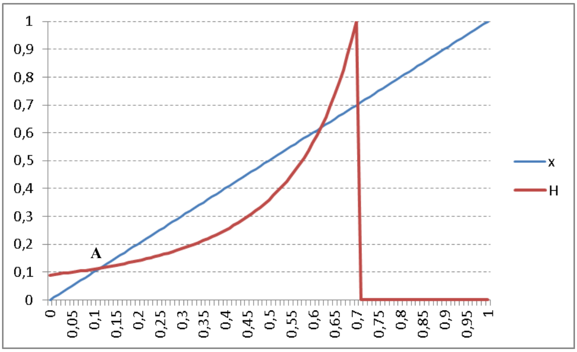

Fig. 6. The right-hand side of the expression (12) in Example 2

Under any initial conditions $x^0 \in [0; 1]$, the dynamical system (12) has an equilibrium, see point A in Fig. 6. •

4. Model of individual and collective penalties

Let us modify the goal function (9) as follows:

$$(13)\ f_i(y) = H_i - c_i\, y_i - \gamma\, H_i\, \frac{1-y_i}{N-Y} - (1 - y_i)\, \delta\, H_i\, I(Y < \hat{Y}),$$

where $\delta \ge 0$ and the last term reflects agent's losses due to bad environmental conditions (it can improve these conditions by individual actions or collective actions of other agents).

By evaluating the best response of agent i, we easily obtain

$$(14)\ y_i = \begin{cases} 1, \text{ if } x_{-i} \in [1-\dfrac{\gamma\rho_i}{1-\delta\rho_i}; \hat{x}] \text{ or } x_{-i} \ge \max\{\hat{x}; 1-\gamma\rho_i\}, \\ 0, \text{ otherwise.} \end{cases}$$

The expression (14) implies that the share of agents investing their financial resources in environmental measures satisfies the condition

(15) $x^k = \max\ \{0;\ 1 - F(\dfrac{1-x^{k-1}}{\gamma}) - G(x^{k-1}) + F(\dfrac{1-x^{k-1}}{\gamma+\delta(1-x^{k-1})})\}.$

Here the control problem consists in choosing "the penalty strength" γ and δ (motivational control) and agents' awareness, e.g., about the threshold $\hat{x}$ (informational control), which guarantee the desired dynamics of the system.

Example 3. Set $F(z) = \sqrt{z}$, $G(z) = z^4$, $\gamma = 2$, $\delta = 3$, and $x^0 = 0.7$. The curve of the right-hand side of the expression (15) and the corresponding trajectory are illustrated by Fig. 7 and Fig. 8.

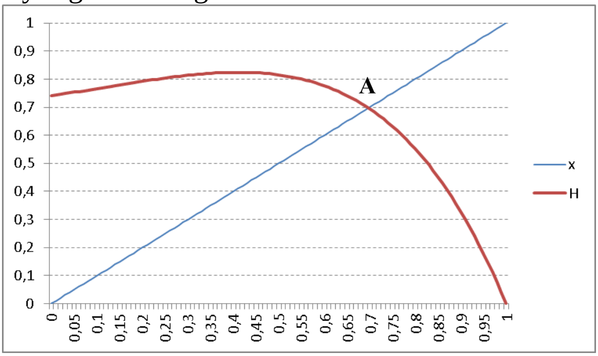

Fig. 7. The right-hand side of the expression (15) in Example 3 under $\gamma = 2$, $\delta = 3$

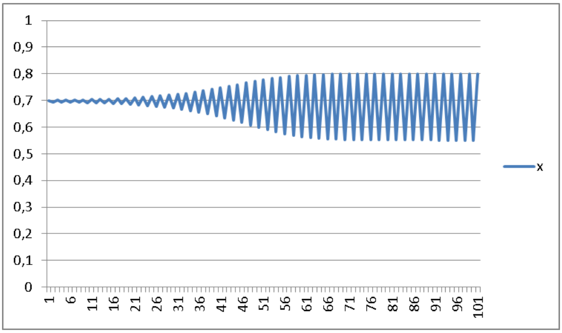

Fig. 8. The trajectory of the system (15) in Example 3 under $\gamma = 2$, $\delta = 3$

Under the parameters of Example 3 and $\gamma = 1$, we observe system stabilization (see Fig. 9), but the equilibrium share of active agents is smaller than in the initial condition.

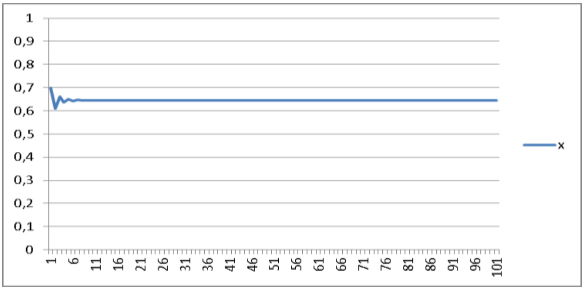

Fig. 9. The trajectory of the system (15) *in Example 3 under* $\gamma = 1$, $\delta = 3$

By choosing $\gamma = 1$, $\delta = 11$, and $\hat{x} = 0.9$ (see Fig. 10), we obtain the system dynamics shown in Fig. 11 (the equilibrium share of active agents increases in comparison with the previous case). •

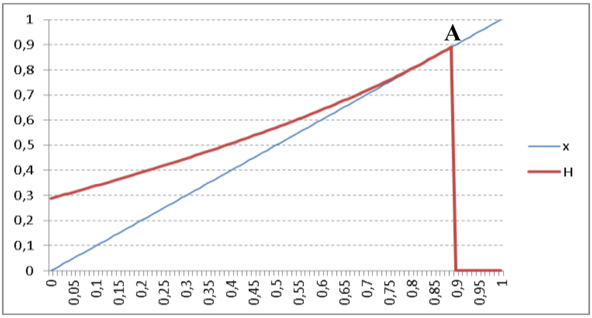

Fig. 10. The right-hand side of the expression (15) *in Example 3 under* $\gamma = 1$, $\delta = 11$, *and* $\hat{x} = 0.9$

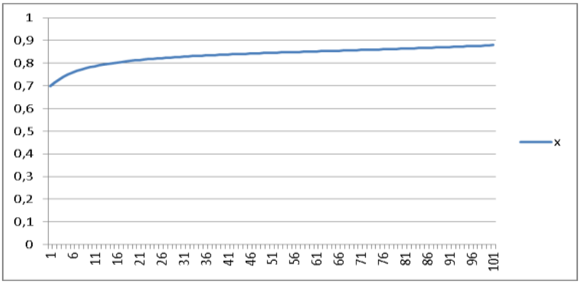

Fig. 11. The trajectory of the system (15) *in Example 3 under* $\gamma = 1$, $\delta = 11$, *and* $\hat{x} = 0.9$

5. Incentive model of environmental protection

Consider the goal function of agent i in the form

(16) $f_i(y) = H_i - c_i\, y_i + y_i\, Q_0\, I(Y \geq \hat{Y}) / Y$,

where the quantity $Q_0 \geq 0$ can be comprehended as an incentive fund for environmental measures. This fund is allocated in equal shares among all agents investing in environmental measures (also see joint financing mechanisms in [18, 20]).

Calculate the best response of agent i:

(17) $y_i = \begin{cases} 1, \text{ if } x_{-i} \in [\hat{x}; c_i / Q], \\ \quad 0, \text{ otherwise.} \end{cases}$

where $Q = Q_0 / n$ means "the specific incentive."

According to (17), the share of agents investing their financial resources in environmental measures meets the condition

(18) $x^k = \max \{0; G(x^{k-1}) - P(Q\, x^{k-1})\}$,

where $P(\cdot)$ is the cost distribution function of the agents.

Here the control problem lies in choosing "the specific incentive" Q and (like in the previous two models) in choosing agents' awareness about the threshold $\hat{x}$ and the value of this threshold.

<u>Example 4.</u> Set $P(z) = z^2$. The curve of the right-hand side of (18) and the corresponding trajectory are presented by Fig. 12 and Fig. 13.

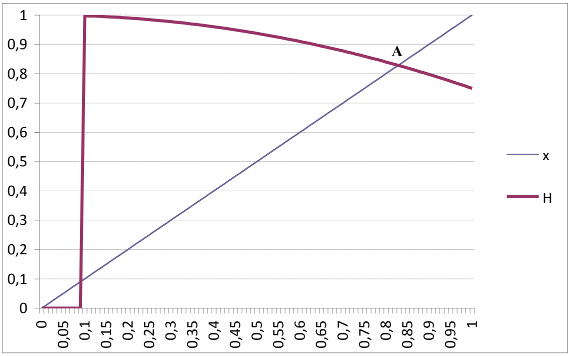

Fig. 12. The right-hand side of the expression (18) *in Example 4 under* $\hat{x}$ = 0.1, Q = 0.5

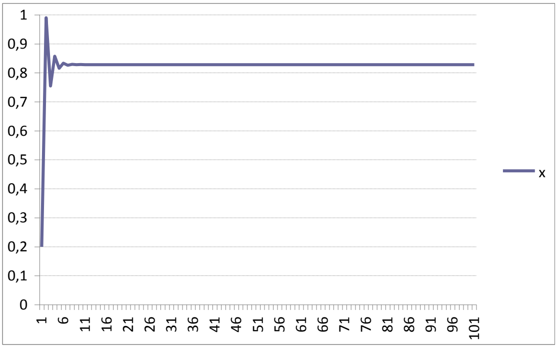

Fig. 13. The trajectory of the system (18) *in Example 4 under* $\hat{x}$ = 0.1, Q = 0.5

As we increase the threshold $\hat{x}$ (i.e., toughen the conditions of funding), this mechanism ceases to be motivating–see Fig. 14.

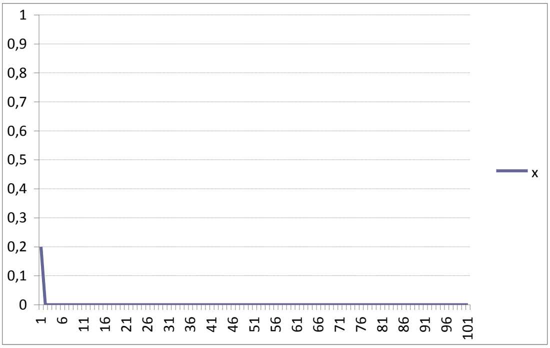

Fig. 14. The trajectory of the system (18) *in Example 4 under* $\hat{x}$ = 0.25, Q = 0.5

Curiously enough from common sense, higher incentives for agents' environmental measures can cause instable behavior of the agents (see Fig. 15) or even demotivate them (see Fig. 16). •

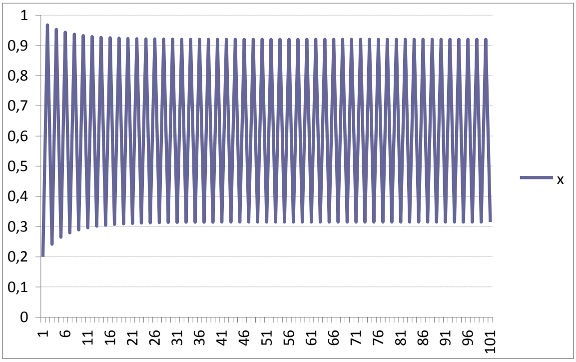

Fig. 15. The trajectory of the system (18) *in Example 4 under* $\hat{x}$ = 0.1, Q = 0.9

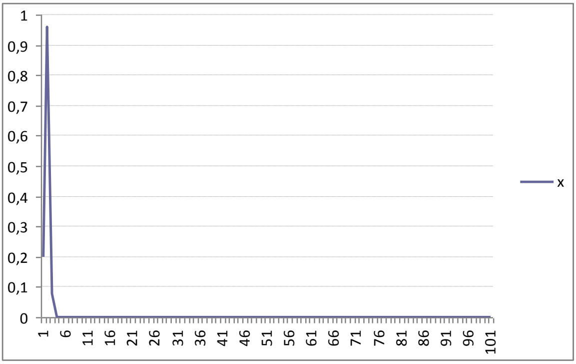

Fig. 16. The trajectory of the system (18) *in Example 4 under* $\hat{x}$ = 0.1, Q = 1

To assess the efficiency of incentive mechanisms of environmental activity, one can use the ratio of "investments" (the equilibrium costs of the agents) and the incentive fund.

And finally, note that it seems interesting to analyze models, where incentive fund depends on the number or share of active agents.

6. Conclusion

In this paper, we have applied the general description of multi-threshold collective behavior to control problems of ecological0economic systems.

The framework of the three studied models of incentives and penalties for environmental protection and restoration allows for proper consideration and exploration of the following phenomena:

- higher “penalty strength” increases the share of agents investing their financial resources in environmental measures;
- uncertainty reduction with respect to the institutional conditions of agents’ functioning increases the share of agents investing their financial resources in environmental measures;
- penalty constraints have to be carefully specified, since otherwise the controlled system may demonstrate instability;
- tougher conditions of agents’ funding for their environmental measures can make this incentive mechanism no more motivating;
- higher incentives of agents (for environmental measures) can cause their instable behavior and even demotivate them.

Generally speaking, we acknowledge that the above models enjoy all advantages of discrete nonlinear dynamical models (the feasibility of reflecting many qualitative effects, simple realization of numerical experiments, and so on), as well as suffer from all their drawbacks (complicated analytical study of equilibria and their uniqueness, system stability and the domains of equilibria attraction, strong dependence of equilibria on model parameters and initial conditions and others).

In the context of control problems, this means the need for maximally accurate identification of controlled objects and inevitable system response simulation (anticipating practical usage of control actions) depending on its parameters and initial conditions.

Regarding the promising lines of theoretical studies, we mention design and analysis of general models of multi-threshold collective behavior.

References

1 Batov A.V., Breer V.V., Novikov D.A., and Rogatkin A.D. Micro- and Macromodels of Social Networks. II. Identification and Simulation Experiments // *Automation and Remote Control*. – 2015 (in press).

2 Breer V.V. Conformal Behavior Models. I. From Philosophy to Math Models; II. Mathematical Models // *Control Sciences*. - 2014. - No. 1. - P. 2–13; - No. 2. - P. 2–17.

3 Breer V.V. Game-Theoretic Models of Collective Conformity Behavior // *Automation and Remote Control*. – 2012. - Vol. 73. - No. 10. - P. 1680-1692.

4 Breer V.V. and Rogatkin A.D. A Probabilistic Model of Threshold Behavior in Multiagent Systems // Automation and Remote Control. - 2015 (in press).

5 Vasin A.A. *Nekooperativnye igry v prirode i obshchestve* (Noncooperative Games in Nature and Society). – Moscow: MAKS Press, 2005. – 412 p. (in Russian).

6 Gorelik V.A. and Kononenko A.F. *Teoretiko-igrovye modeli prinyatiya reshenii v ekologo-ekonomicheskikh sistemakh* (Game-Theoretic Models of Decision-Making in Ecological-Economic Systems). – Moscow: Radio i Svyaz', 1982. – 144 p. (in Russian).

7 Gorstko A.B., Dombrovskii Yu.A., and Surkov F.A. *Modeli upravleniya ekologo-ekonomicheskimi sistemami* (Control Models of Ecological-Economic Systems). – Moscow: Nauka, 1984. – 120 p. (in Russian).

8 *Modelirovanie i upravlenie protsessami regional'nogo razvitiya* (Modeling and Control of Regional Development Processes) / Ed. by S.N. Vasil'ev. – Moscow: Fizmatlit, 2001. – 432 p. (in Russian).

9 Petrosjan L.A. and Zakharov V.V. *Vvedenie v matematicheskuyu ekologiyu* (Introduction to Mathematical Ecology). – Leningrad: Izd. Leningrad. Gos. Univ., 1986. – 224 p. (in Russian).

10 Ougolnitsky G.A. *Upravlenie ekologo-ekonomicheskimi sistemami* (Control of Ecological-Economic Systems). – Moscow: Vuzovskaya Kniga, 2004. – 132 p. (in Russian).

11 Breer V.V. A Game-Theoretic Model of Non-Anonymous Threshold Conformity Behavior // *Automation and Remote Control*. - 2012. - Vol. 73. - No. 7. - P. 1256–1264.

12 Breer V.V. and Novikov D.A. Models of Mob Control // *Automation and Remote Control*. - 2013. - Vol. 74. - No 12. - P. 2143–2154.

13 Broom M. and Rychtar J. *Game-theoretical Models in Biology*. - Leiden: CRC, 2013. – 520 p.

14 Burkov V.N., Novikov D.A., and Shchepkin A.V. *Control Mechanisms for Ecological-Economic Systems*. – Berlin: Springer, 2015. – 174 p.

15 Granovetter M. Threshold Models of Collective Behavior // *The American Journal of Sociology*. – 1978. – Vol. 83. No 6. – P. 1420–1443.

16 Janssen M. and Ostrom E. Governing Social-Ecological Systems / *Handbook of Computational Economics*. Vol. 2. Ed. by Leigh Tesfatsion L. and Judd K. – Oxford: Elseiver, 2006. P. 1466–1509.

17 Levy M. *Market Efficiency, the Pareto Wealth Distribution and the Levy Distribution of Stock Returns*. - Jerusalem: Hebrew University, 2001. – 52 p.

18 *Mechanism Design and Management: Mathematical Methods for Smart Organizations* / Ed. by Prof. D.A. Novikov. – N.Y.: Nova Science Publishers, 2013. – 204 p.

19 Novikov D.A. Problems of Stimulating Pareto-agent // *Automation and Remote Control*. - 2007. - Vol. 68. - No 1. – P. 124–132.

20 Novikov D.A. *Theory of Control in Organizations*. – N.Y.: Nova Scientific Publishing, 2013. – 341 p.

21 Ougolnitsky G. *Sustainable Management*. – N.Y.: Nova Scientific Publishing, 2012. – 287 p.

22 Schelling T. *Micromotives and Macrobehavior*. – N.Y.: WW Norton & Co, 1978. – 272 p.

23 *Theory and Implementation of Economic Models for Sustainable Development* / Ed. by van den Bergh J. and Howkes M. – Dordrecht: Kluwer Academic Publishers, 2010. – 328 p.

24 Whitelaw K. *Environmental Systems Handbook*. – Oxford: Elsevier, 2004. – 237 p.