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MATHEMATICAL MODELS OF INFORMATIONAL AND STRATEGIC REFLEXION: A SURVEY

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The paper is dedicated to a survey (in the framework of game theory and theory of collective behavior) of modern approaches to mathematical modeling of reflexive games and reflexive processes in control.

REFLEXION. A fundamental property of human entity lies in the following. In addition to natural ("objective") reality, there exists its image in human minds. Furthermore, an inevitable gap (mismatch) takes place between the latter and the former. In the sequel, the described image will be called a part of reflexive reality. Traditionally, purposeful study of this phenomenon relates to the term "reflexion." The term reflexion (from Latin reflex 'bent back'; was first suggested by J. Locke) means [62]:

- a principle of human thinking, guiding humans towards comprehension and perception of one's own forms and premises;
- subjective consideration of a knowledge, critical analysis of its content and cognition methods;
- the activity of self-actualization, revealing the internal structure and specifics of spiritual world of a human.

To elucidate the whole essence of reflexion, let us consider the case of a single subject. He/she possesses certain beliefs about natural reality; however, a subject may perform reflexion (construct images) with respect to these beliefs (thus, generating new beliefs). Generally, this process is infinite and results in formation of reflexive reality. The reflexion of a subject with respect to his/her own beliefs of reality, principles of his/her activity, etc., is said to be self-reflexion or reflexion of the first kind. We emphasize that most social research works concentrate on self-reflexion. In philosophy, self-reflexion represents the process of individual's thinking about beliefs in his/her own mind [57]. Reflexion of the second kind takes place with respect to other subjects (includes beliefs of a subject about possible beliefs, decision principles and self-reflexion of other subjects).

REFLEXION AND CONTROL. *Control* is an element, a function of organized systems of different nature (biological, social, technical, etc.), preserving their definite structure, sustaining their mode of activity and implementing the program or goal of their activity; control is a purposeful impact exerted on a controlled system to ensure its required behavior [61].

Assume there is a control subject (a principal) and a controlled system (control object–in terminology of technical systems–or a controlled subject). The state of a controlled system depends on external disturbances, control actions applied by a principal and possibly on actions performed by the controlled system (if the latter represents an active subject), see Fig. 1. The principal's problem lies in choosing control actions (see the thick line in Fig. 1) to ensure the required behavior of a controlled system taking into account information on external disturbances (see the dashed line in Fig. 1).

The so-called input-output structure of a control system (Fig. 1) is typical for control theory dealing with control problems in systems of different nature. The presence of feedback (see the double line in Fig. 1) which provides a principal with information on the state of a controlled system is the key (but not compulsory!) property of a control system. Some researchers interpret feedback as reflexion (as an image of the controlled system's state in the "mind" of a control subject). This forms the first aspect of interrelation between control and reflexion.

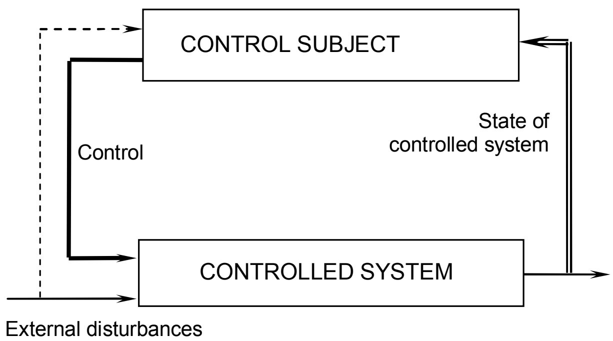

Fig. 1. The structure of a control system

A series of scientific directions investigate the interaction and activity of a control subject and controlled system. Control science (or *control theory* in the terminology of corresponding experts) mostly focuses on the interaction between a control subject and controlled system. *Control methodology* [59] is the theory of organizing of control activity, i.e., the activity performed by a control subject. We emphasize that *activity* can be mentioned only with respect to active subjects (e.g., a human being, a group, a collective). In the case of passive (e.g., technical) systems, the term "functioning" is used instead. In the sequel, we believe that a control subject and controlled system appear active (otherwise, there is a clear provision for the opposite). Hence, **each of them may perform (at least) self-reflexion**, constructing "images" of the process, organization principles and results of his/her own activity. This is the second aspect of interrelation between control and reflexion.

Searching for *optimal control* (i.e., the most efficient admissible control) requires control subject's ability of predicting controlled system's response to certain control actions. One of prerequisites is a model of a controlled system. Generally speaking, a *model* is an image of a certain system; an analog (a scheme, a structure or a sign system) of a certain fragment of the natural or social reality, a "substitute" for the original in cognition process and practice. A model can be considered as an image of a controlled system in the mind of a control subject. **Modeling** (as a process of "reflecting," i.e., constructing this image) **can be viewed as reflexion**. Furthermore, a controlled system may predict and assess the activity performed by a control subject. And so, we obtain the third aspect of interrelation between control and reflexion.

The fourth aspect lies in the following. **A control subject or controlled system performs reflexion with respect to external subjects and objects**, phenomena or processes, their properties and laws of activity/functioning. For instance, the matter concerns an external environment (for a control subject), an external environment and/or other elements of a controlled system (for a fixed element of a controlled system). Indeed, suppose that a controlled system includes several active agents; each of them may perform reflexion with respect to the others. Exactly this aspect–mutual reflexion of controlled subjects–is discussed in game-theoretical models.

Of crucial importance here is that **the process and/or result of reflexion can be controlled**, i.e., can represent a component of controlled system's activity, being modified by a control subject for a definite goal. Precisely this relationship between control and reflexion enables informational control and reflexive control, considered below.

GAME THEORY. Formal (mathematical) models of human behavior have been constructed and studied for over last 150 years. Gradually, these models find wider application in control theory, economics, psychology, sociology, etc., as well as in practical problems. In the sequel, we will understand a *game* as the interaction of subjects with noncoinciding interests. Still, an alternative interpretation treats a game as a type of unproductive activity whose motive consists not in the corresponding results, but in the process of activity itself (see [36, 57], where the notion of a game is assigned a broader sense).

Game theory represents a branch of applied mathematics, which analyzes models of decision making in the conditions of noncoinciding interests of opponents (*players*); each player strives for influencing the situation in his/her favor [27, 52]. In what follows, a decision-maker (a player) is called an *agent*. The major task of game theory is describing the interaction among several agents with noncoinciding interests, where the results of agent's activity (payoff, utility, etc.) generally depend on actions of all agents. Such description yields a forecast of a rational and "stable" outcome of the game–the so-called *game solution* (*equilibrium*).

Describing a *game* means specifying the following parameters:

- *a set of agents*;
- *preferences of agents* (relationships between payoffs and actions). Each agent is supposed to strive for maximizing his/her payoff (and so, the behavior of each agent appears purposeful);
- *a set of feasible actions of agents*;
- *awareness of agents* (information on essential parameters, being available to agents at the moment of their choice);
- *sequence of moves* (the sequence of obtaining information and choosing actions).

The above parameters define a game; unfortunately, they are insufficient for forecasting its outcome, i.e., a solution (or an equilibrium) of the game–the set of rational and stable actions of agents. Nowadays, game theory suggests no universal concept of equilibria. By adopting different assumptions regarding principles of agent's decision making, one can construct different solutions. Thus, designing an equilibrium concept forms a basic problem for any game-theoretic research; this book does not represent an exception, as well. Reflexive games are defined as a direct interaction among agents, where they make decisions based on hierarchies of their beliefs. In other words, awareness of agents is extremely important.

THE ROLE OF AWARENESS. COMMON KNOWLEDGE. In game theory, psychology, distributed systems and other fields of science (see the overviews in [26, 51]), one should consider not only agents' *beliefs* about essential parameters, but also their beliefs about the beliefs of other agents, etc. The set of such beliefs is called the *hierarchy of beliefs*. We will model it using the tree of awareness structure of a reflexive game (see below). In other words, situations of interactive decision making (modeled in game theory) require that each agent "forecasts" opponents' behavior prior to his/her choice. And so, each agent should possess definite beliefs about the view of the game by his/her opponents. On the other hand, opponents should do the same. Consequently, the uncertainty regarding the game to-be-played generates an infinite hierarchy of beliefs of game participants.

A special case of awareness concerns *common knowledge* when beliefs of all orders coincide. A rigorous definition of common knowledge was introduced in [45]. Notably, common knowledge is a fact with the following properties:

1) all agents know it;
2) all agents know 1;
3) all agents know 2 and so on–*ad infinitum*.

The formal model of common knowledge was originally proposed in [1]. Later on, many investigators refined and redeveloped it – see surveys and references in [3, 22, 23, 24, 34, 46, 62, 67, etc].

The present paper is almost completely dedicated to models of agents' awareness in <u>game theory</u> (*viz*., hierarchies of beliefs and common knowledge). Thus, we give several references demonstrating the role of common knowledge in different fields of science–philosophy, psychology, etc. (see also the overview in [21]). In philosophy, common knowledge has been studied in *convention* analysis [45, 74]. In psychology, one would face the notion of *discourse* (from Latin *discursus* 'argument'). It means human thinking in words, being mediated by past experience; discourse acts as the process of connected logical reasoning, where a next idea stems from the previous one. The importance of common knowledge in discourse comprehension has been explored in [15, 21]. Mutual awareness of agents turns out significant in distributed computer systems [22, 24, 30], artificial intelligence [29, 48] and other fields.

Game theory often assumes that all[1] parameters of a game are a *common knowledge*. Such assumption corresponds to the *objective description of a game* and enables addressing the *Nash equilibrium*[2]concept [55] as a forecasted outcome of a noncooperative game (a game, where agents do not agree about coalitions, data exchange, joint actions, redistribution of payoffs, etc.). Thus, the assumption regarding common knowledge allows claiming that all agents know which game they play and that their beliefs about the game coincide.

Generally, each agent may possess individual beliefs about parameters of a game. And so, each belief corresponds to a *subjective description of the game* [27] (see also modern models of awareness in [16, 25, 33, 66]). Consequently, agents participate in the game, having no objective views of it or interpreting this game in different ways (rules, goals, the roles and awareness of opponents, etc.). Unfortunately, still no universal approaches have been proposed for equilibria design under insufficient common knowledge.

On the other part, within the "reflexive tradition" of the humanities, the surrounding world of each agent includes the rest agents; moreover, beliefs about other agents get reflected during the process of reflexion (in particular, variations of beliefs may result from nonidentical awareness). However, researchers have not succeeded in deriving constructive formal outcomes in this field to date.

Hence, an urgent problem lies in designing and analyzing mathematical models of games, where agents' awareness is not a common knowledge and agents make decisions based on hierarchies of their beliefs. Such class of games is called **reflexive games** [13, 63, 64]. We will provide a formal definition later.

The term "reflexive games" was introduced by V. Lefebvre in 1965, see [41]. However, the cited work and his other publications [40, 42–44] represented qualitative discussions of reflexion effects in interaction among subjects (actually, no general concept of solution was suggested for this class of games). Similar remarks apply to [20, 28, 68, 72], where a series of special cases of players' awareness was studied. The monograph [64] concentrated on systematical treatment of reflexive games and an endeavor of constructing a uniform equilibrium concept for these games.

According to game theory and reflexive models of decision making, it seems reasonable to distinguish between strategic reflexion and informational reflexion. *Informational reflexion* is the process and result of agent's thinking about (a) the values of uncertain parameters and (b) what his/her opponents (other agents) know about these values. Here the "game" component actually disappears–an agent makes no decisions. *Strategic reflexion* is the process and result of agent's thinking about which decision making principles his/her opponents (other agents) employ under the awareness assigned by him/her via informational reflexion. Therefore, informational reflexion often relates to insufficient mutual awareness, and its result serves for decision making (including informational reflexion). Strategic reflexion takes place even in the case of complete awareness, precessing agent's choice of an action. In other words, informational and strategic reflexion can be studied independently, but the both occur in the case of incomplete or insufficient awareness.

GENERAL APPROACHES TO THE DESCRIPTION OF INFORMATIONAL AND STRATEGIC REFLEXION. According to [2, 3, 32], there are two different approaches to the description of awareness structures, *viz*., *syntactic* and *semantic* ones. Recall that syntactics means syntax of sign systems, i.e., the structure of sign combinations and rules of their formation, "translation" and interpretation irrespective of their values and functions of sign systems. Semantics studies sign systems as tools of meaning expression; here the basic subject lies in interpretations of signs and sign combinations. Foundations of these approaches were laid in mathematical logic [35, 39].

Within the framework of syntactic approach, an hierarchy of beliefs is described explicitly. Suppose that beliefs are defined by a probability distribution. Then hierarchies of beliefs (at a certain level) correspond to distributions on the product of the set of states of nature and distributions reflecting beliefs of preceding levels [50]. An alternative is using "logic formulas"–rules of transforming elements of an initial set based on logic operations and operators such as "player i believes the probability of event … is not smaller than α" [32, 77]. A knowledge is modeled by propositions (formulas) constructed according to certain syntactic rules.

According to semantic approach, beliefs of agents are defined by probability distributions on the set of states of nature. Hierarchies of beliefs get generated only by virtue of these distributions. In the elementary (deterministic) case, a knowledge represents the set Θ of feasible values of an uncertain parameter and different partitions $\{P_i\}_{i \in N}$ of this set. An element of the partition P_i containing $\theta \in \Theta$ forms the knowledge of agent i, namely, the set of values of the uncertain parameter, being indistinguishable for this agent under a known fact θ [1, 3]. The correspondence (or "equivalence") between syntactic and semantic approaches was established in [2, 67] and other works. We also cite experimental research on hierarchies of beliefs [7, 53, 70]; see the surveys in [60, 75].

[1] If the initial model incorporates uncertain factors, specific procedures of uncertainty elimination are involved to obtain a deterministic model.
[2] An agents' action vector is a Nash equilibrium if none of them benefits by unilateral deviation from it (provided that the rest agents choose the corresponding components of the Nash equilibrium). A more rigorous definition could be found below.

The above overview points at two existing “extremes.” The first one lies in common knowledge. Here J. Harsanyi’s merits [31] are (a) reducing all information on an agent (determining the latter’s behavior) to a single characteristic–agent’s type–and (b) constructing a Bayes-Nash equilibrium by hypothesizing that the probability distribution of types is a common knowledge. The second “extreme” relates to infinite hierarchy of compatible or incompatible beliefs. (For an example, see the structure discussed in [50]. On the one hand, it describes all possible Bayesian games and all possible hierarchies of beliefs. On the other hand, it appears very general and, consequently, very cumbersome, thus interfering with constructive statement and solution of specific problems).

Most research on awareness seeks to answer the following question. When does an hierarchy of agents’ beliefs describe a common knowledge and/or reflect adequately their awareness? [5, 21]. The dependence of game solutions on a finite hierarchy of compatible or incompatible beliefs of agents (the whole range between the above “extremes”) has been studied in [62, 64].

THEORY OF COLLECTIVE BEHAVIOR. Traditionally, game-theoretic models and/or models of collective decision making utilize one of two assumptions regarding mutual awareness of agents [60]. The first one implies that all essential information and decision principles adopted by agents are known to all agents, all agents know this fact and so on (such reasoning could be infinite). Actually, this is the concept of a *common knowledge*, which serves, e.g., in constructing a Nash equilibrium. The second assumption claims that each agent (according to his/her awareness) follows a certain procedure of individual decision making and has “almost no idea” of the knowledge and behavior of the rest agents. The first approach appears canonical in *game theory*, while the second approach has become popular in models of *collective behavior*. Yet, a variety of intermediate situations exists between these “extreme cases.” Imagine that informational reflexion takes no place–a common knowledge on essential external parameters is observed. Let an agent have performed an act of *strategic reflexion*, i.e., an attempt to predict the behavior of other agents (not their awareness but decision principles). This agent chooses his/her actions using the forecast (we believe he/she possesses reflexion rank 1). Another agent (with reflexion rank 2) possibly knows about the existence of agents having reflexion rank 1. Consequently, such agent endeavors to predict their behavior, as well. Again, this line of reasoning could be infinite. A series of questions arises immediately. How does the behavior of a collective of agents depend on their distribution by reflexion rank (the number of agents with a specific rank in a collective)? Suppose that the shares of reflexing agents can be controlled. What are the optimal values of these shares? Here optimality is “measured” in terms of some criterion defined on the set of agents’ actions.

Classic game-theoretic models proceed from the following. In a normal form game, agents choose Nash equilibrium actions. However, investigations in the field of *experimental economics* indicate this not always the case (e.g., see [73] and the overview [78]). The divergence between actual behavior and theoretical expectations has several explanations:

– limited cognitive capabilities of agents [37] (decentralized evaluation of a Nash equilibrium represents a cumbersome computational problem [56]). Furthermore, sometimes Nash equilibria provide no adequate description to the real behavior of agents in experimental single stage games (agents have not enough time for “correcting” their wrong beliefs about essential parameters of a game [4]). For instance, D. Bernheim’s concept of rationalizable strategies requires unlimited rationality from agents (their high cognitive capabilities);

– agent’s full confidence in that all the opponents would evaluate a Nash equilibrium;

– incomplete awareness;

– the presence of several equilibria.

Therefore, there exist at least two foundations (“theoretical” and “experimental” ones) for considering models of collective behavior of agents with different reflexion ranks.

In contrast to game theory, the *theory of collective behavior* analyzes the behavior dynamics of rational agents under rather weak assumptions regarding their awareness. For instance, far from always agents need a common knowledge about the set of agents, sets of feasible actions and goal functions of opponents. Alternatively, agents may not predict the behavior of their opponents (as in game theory). Moreover, making decisions, agents may “know nothing about the existence of” specific agents or possess aggregated information about them.

The most widespread model of collective behavior dynamics is the *model of indicator behavior* (see references in [60]). The essence of the model consists in the following. Suppose that at instant t each agent observes the actions of all agents $\{x_i^{t-1}\}_{i\in N}$ that have been chosen at the preceding instant $t-1$, $t = 1, 2, \ldots$ The initial action vector $x^0 = (x_1^0, \ldots, x_n^0)$ is assumed known.

Each agent can evaluate his/her *current goal*–an action maximizing his/her goal function provided that at a current instant all agents choose the same actions as at the previous instant:

(1) $w_i(x_{-i}^{t-1}) = \arg \max\limits_{y\in\Re^1} F_i(y, x_{-i}^{t-1})$, $t = 1, 2, \ldots, i \in N$.

According to the hypothesis of indicator behavior, at each instant an agent makes a “step” from his/her previous action to the current goal:

(2) $x_i^t = x_i^{t-1} + \gamma_i^t\, [w_i(x_{-i}^{t-1}) - x_i^{t-1}]$, $i \in N$, $t = 1, 2, \ldots$,

where $\gamma_i^t \in [0; 1]$ designate “the values of steps.” For convenience, such collective behavior can be called “optimization behavior” (thus, we emphasize its difference from play behavior). The approaches adopted by the theory of collective behavior and game theory agree in the following sense. The both study the behavior of rational agents, while game equilibria generally represent equilibria for dynamic procedures of collective behavior. For instance, the Nash equilibrium specifies an equilibrium for the dynamics (2) of collective behavior.

To make the picture complete, note one more aspect, as well. The theory of collective behavior proposes another approach (going beyond the scope of this book), namely, *evolutionary game theory* [76]. This science studies the behavior of large homogeneous groups (populations) of individuals in typical repeated conflicts; each strategy is applied by a set of players, whereas a corresponding goal function characterizes the success of specific strategies (instead of specific participants of such interaction).

Thus, game theory often employs maximal assumptions regarding agents' awareness (e.g., the hypothesis of existing common knowledge), while the theory of collective behavior involves the minimal assumptions. The intermediate position belongs to reflexive models. And so, let us discuss the role of (informational and strategic) reflexion in decision making by agents.

REFLEXION IN GAME THEORY AND MODELS OF COLLECTIVE BEHAVIOR: THE STRUCTURE OF PROBLEM DOMAIN. Game theory and the theory of collective behavior analyze interaction models for rational agents. Approaches and results of these theories can be considered at three interconnected epistemological levels (that correspond to different functions of modeling [57])–see Fig. 2 [62]:

– phenomenological level, where a model aims at describing and/or explaining the behavior of a system (a collective of agents);

– predictive level (the aim is forecasting the system behavior);

– normative level (the aim is ensuring a required system behavior).

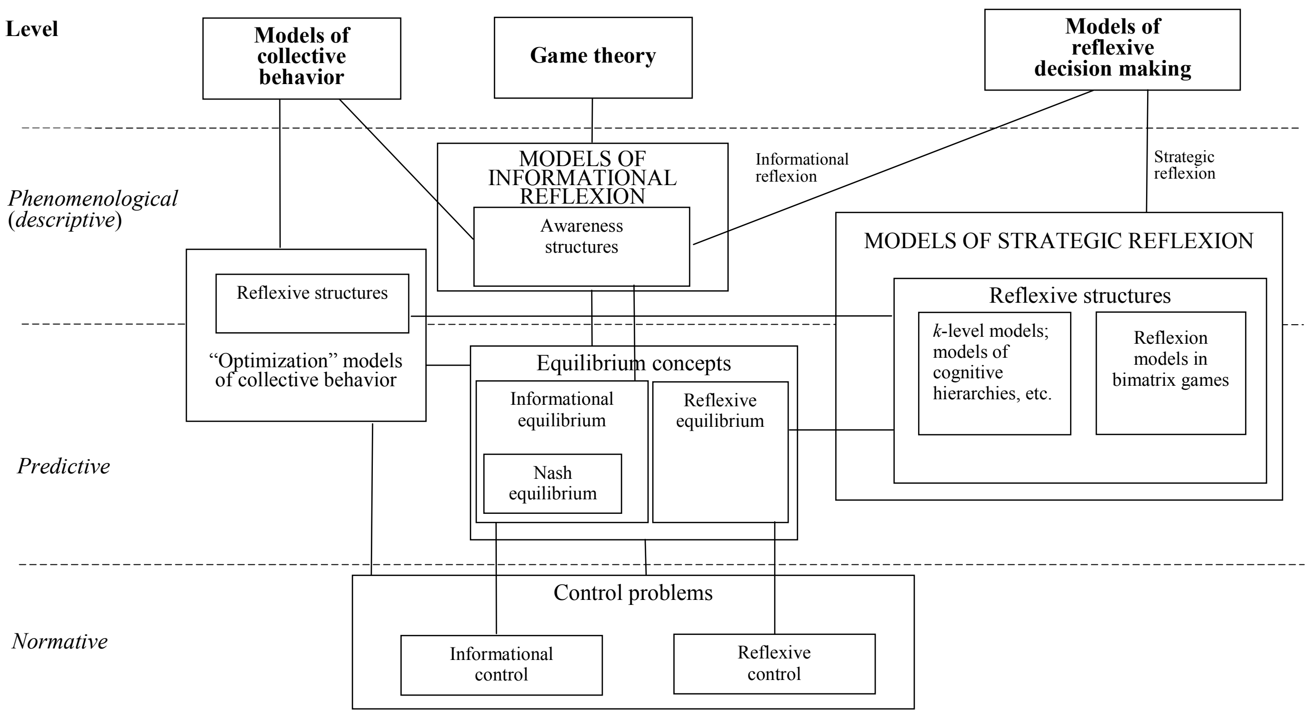

Fig. 2. Descriptive and normative models of informational and strategic reflexion

In game theory, a common scheme consists in (1) describing the "model of a game" (phenomenological level), (2) choosing an equilibrium concept defining the stable outcome of a game (predictive level) and (3) stating a certain control problem–find values of controlled "game parameters" implementing a required equilibrium (normative level). An interested reader would find the corresponding illustration in Fig. 2.

Taking into account *informational reflexion* leads to the necessity of constructing and analyzing awareness structures. This enables defining an informational equilibrium, as well as posing and solving informational control problems–see Fig. 2. Taking into account strategic reflexion generates a similar chain marked by heavy lines in Fig. 2: "models of strategic reflexion" – "reflexive structure" – "reflexive equilibrium" – "reflexive control."

A comparison of approaches to modeling of informational and strategic reflexion is given by Table 1.

Table 1. Modeling of informational and strategic reflexion: a comparison of approaches

Parameter	Informational reflexion	Strategic reflexion
Model of a "game"	Awareness structure	Reflexive structure
Equilibrium	Informational equilibrium	Reflexive equilibrium
Control	Informational control	Reflexive control

AWARENESS STRUCTURE AND INFORMATIONAL EQUILIBRIUM. Consider the set of agents: $N = \{1, 2,..., n\}$. Denote by $\theta \in \Theta$ the uncertain parameter (we believe that the set Θ is a common knowledge for all agents). The ***awareness structure*** I_i of agent i includes the following elements. First, the belief of agent i about the parameter θ; denote it by θ_i, $\theta_i \in \Theta$. Second, the beliefs of agent i about the beliefs of the other agents about the parameter θ; denote them by θ_{ij}, $\theta_{ij} \in \Theta, j \in N$. Third, the beliefs of agent i about the beliefs of agent j about the belief of agent k; denote them by θ_{ijk}, $\theta_{ijk} \in \Theta$, $j, k \in N$. And so on (evidently, this reasoning is generally infinite). In the sequel, we employ the term "*awareness structure*," which is a synonym of "*informational structure"* and "*hierarchy of beliefs*." Therefore, the awareness structure I_i of agent i is specified by the set of values $\theta_{ij_1...j_l}$, where l runs over the set of nonnegative integer numbers, $j_1, \ldots, j_l \in N$, while $\theta_{i_1...i_l} \in \Theta$.

The *awareness structure I of the whole game* is defined in a similar manner; in particular, the set of the values $\theta_{i_1...i_l}$ is employed, with l running over the set of nonnegative integer numbers, $j_1, \ldots, j_l \in N$, and $\theta_{ij_1...j_l} \in \Theta$. We emphasize that the agents are not aware of the whole structure I; each of them knows only a substructure I_i. Thus, an awareness structure is an infinite n-tree; the corresponding nodes of the tree describe specific awareness of real agents from the set N, and also phantom agents (complex reflexions of real agents in the mind of their opponents).

A ***reflexive game*** Γ_I is a game defined by the following tuple:

(3) $\Gamma_I = \{N, (X_i)_{i \in N}, f_i(\cdot)_{i \in N}, I\}$,

where N stands for a set of real agents, X_i means a set of feasible actions of agent i, $f_i(\cdot)$: $\Theta \times X' \to \Re^1$ is his/her goal function ($i \in N$); Θ indicates a set of feasible values of the uncertain parameter and I designates the awareness structure.

Therefore, a reflexive game generalizes the notion of a normal-form game (determined by the tuple $\{N, (X_i)_{i \in N}, f_i(\cdot)_{i \in N}\}$) to the case when agents' awareness is reflected by an hierarchy of their beliefs (i.e., the awareness structure I). Within the framework of the accepted definition, a "classical" normal-form game is a special case of a reflexive game (a game under a common knowledge among the agents). Consider the "extreme" case when the state of nature appears a common knowledge; for a reflexive game, the solution concept (proposed in this book based on an informational equilibrium, see below) turns out equivalent to the Nash equilibrium concept.

To proceed and formulate a series of definitions and properties, we introduce the following notation:

Σ_+ stands for a set of finite sequences of indexes belonging to N;

Σ is the sum of Σ_+ and the empty sequence;

$|\sigma|$ indicates the number of indexes in the sequence $\sigma \in \Sigma$ (for the empty sequence, it equals zero); this parameter is known as the length of an index sequence.

Imagine θ_i represents the belief of agent i about the uncertain parameter, while θ_{ii} means the belief of agent i about his/her own belief. It seems then natural that $\theta_{ii} = \theta_i$. In other words, agent i is well-informed on his/her own beliefs. Moreover, he/she assumes that the rest agents possess the same property. Formally, this means that the *axiom of self-awareness* is accepted: $\forall\ i \in N$, $\forall\ \tau, \sigma \in \Sigma$: $\theta_{\tau ii\sigma} = \theta_{\tau i\sigma}$. In particular, being aware of θ_τ for all $\tau \in \Sigma_+$ such that $|\tau| = \gamma$, an agent may explicitly evaluate θ_τ for all $\tau \in \Sigma_+$ with $|\tau| < \gamma$.

In addition to the awareness structures I_i ($i \in N$), one may also analyze the awareness structures I_{ij} (i.e., the awareness of agent j according to the belief of agent i), I_{ijk}, and so on. Let us identify the awareness structure with the agent being characterized by it. In this case, one may claim that n *real* agents (*i-agents*, where $i \in N$) having the awareness structures I_i also play with ***phantom agents*** (*τ-agents*, where $\tau \in \Sigma_+$, $|\tau| \geq 2$) having the awareness structures $I_\tau = \{\theta_{\tau\sigma}\}$, $\sigma \in \Sigma$. It should be emphasized that phantom agents exist merely in the minds of real agents; still, they have an impact on their actions; these aspects will be discussed below.

Assume that the awareness structure I of a game is given; this means that the awareness structures are also defined for all (real and phantom) agents. Within the framework of the hypothesis of rational behavior, the choice of an action x_τ performed by a τ-agent is described by his/her awareness structure I_τ. Hence, the mentioned structure being available, one may model agent's reasoning and evaluate his/her action. On the other hand, while choosing his/her action, the agent models actions of the rest agents (i.e., performs reflexion). Therefore, estimating the game outcome, we should account for the actions of real and phantom agents.

A set of actions x_τ^*, $\tau \in \Sigma_+$, is called an ***informational equilibrium***, if the following conditions are met:

1. the awareness structure I possesses finite complexity ν [13];
2. $\forall \lambda, \mu \in \Sigma : I_{\lambda i} = I_{\mu i} \Rightarrow x_{\lambda i}^* = x_{\mu i}^*$;
3. $\forall\ i \in N$, $\forall\ \sigma \in \Sigma$:

(4) $x_{\sigma i}^* \in \operatorname{Arg}\max\limits_{x_i \in X_i} f_i(\theta_{\sigma i}, x_{\sigma i1}^*, \ldots, x_{\sigma i, i-1}^*, x_i, x_{\sigma i, i+1}^* \ldots, x_{\sigma i, n}^*)$.

Here Condition 1 claims that a reflexive game involves a finite number of real and phantom agents (what happens when this assumption is rejected, is dicussed in [9]). Condition 2 expresses the requirement that the agents with an identical awareness choose identical actions. Finally, Condition 3 reflects rational behavior of agents–each agent strives for maximizing the individual goal function via a proper choice of his/her action. For this, an agent substitutes actions of the opponents into his/her goal function; the actions are rational in the view of the considered agent (according to the available beliefs of the rest agents).

The "classical" concept of a Nash equilibrium is remarkable for its self-sustained nature. Notably, assume that a repeated game takes place and all agents (except agent i) choose the same equilibrium actions. Then agent i benefits nothing by deviating from his/her equilibrium action; evidently, this feature is directly related to the following. Beliefs of all agents about reality are adequate, i.e., the state of nature appears a common knowledge. Generally speaking, the situation may change in the case of an informational equilibrium. Indeed, after a single play of the game some agents (or even all of them) may observe an unexpected outcome due to an inconsistent belief about the state of nature (or due to an inadequate awareness of opponents' beliefs). Anyway, the self-sustained nature of the equilibrium is violated; actions of agents may change as the game is repeated. Informational equilibrium is *stable* [65], if each agent (real or phantom) observes exactly the expected result (in this case agents' awareness does not change). Some models of awareness dynamics are considered in [12].

INFORMATIONAL CONTROL. The model of ***informational control*** (purposeful impact on agents' awareness to form informational structure which leads to the desired informational equilibrium) includes an agent (or several agents) and a principal. Each agent is characterized by the cycle "awareness of the agent → action of the agent → result observed by the agent → awareness of the agent." Generally speaking, these components vary for different agents. At the same time, the cycle could be viewed common for the whole controlled subsystem (i.e., for the complete set of agents). This feature is indicated by the word "*Agent*(*s*)" in Fig. 3. The interaction between an agent (agents) and the principal is characterized by the following elements:

– an informational impact of the principal, which forms a certain awareness of an agent (agents). It seems possible to study the principal's influence on the outcome observed by an agent (agents), see the chain "principal → observed outcome" in Fig. 3;

– an actual outcome of the agent's action (or agents' actions), which has an impact on the preferences of the principal.

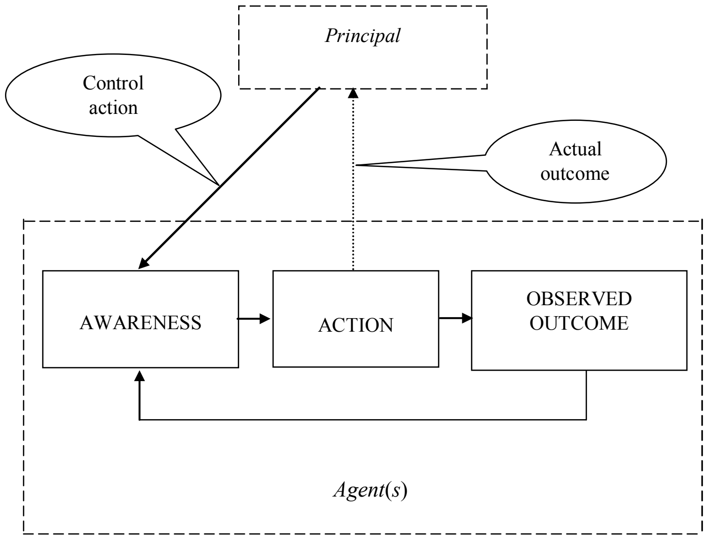

Fig. 3. The model of informational control

Implementing informational control, the principal (as usual) strives to maximize his/her utility. Assume the principal can form any awareness structure from a certain feasible set. The problem of informational control may be posed as follows. Find an awareness structure from the set of feasible structures, which maximizes the principal's utility in a corresponding informational equilibrium (perhaps, taking into account the principal's costs to form such an awareness structure).

Define the following objects: the set $\Psi_X(I) \subseteq X'$ of the action vectors of real agents, representing equilibria under the awareness structure I; and the set $\Psi_I(x)$ of awareness structures, making the action vector x of real agents an equilibrium (solution to the inverse problem).

Let us give a formal statement to the control problem. Assume that the goal function of the principal, $\Phi(x, I)$, is defined on a set of real agents' actions and awareness structures. Next, suppose that the principal can form any awareness structure from a certain set $\mathfrak{I}'$. Under the awareness structure $I \in \mathfrak{I}'$, the action vector of real agents is an element of the set of equilibrium vectors $\Psi_X(I)$. We emphasize that the set $\Psi_X(I)$ may be empty; in the case of a missed equilibrium, the principal cannot predict the outcome of a game. To avoid this problem, introduce the set of feasible structures leading to the non-empty set of equilibria: $\mathfrak{I} = \{I \in \mathfrak{I}' \mid \Psi_X(I) \neq \varnothing\}$.

Imagine that, under the specified awareness structure $I \in \mathfrak{I}$, the set of equilibrium vectors $\Psi_X(I)$ includes (at least) two elements. As a rule, one of the following assumptions is then adopted [61]:

1) *the hypothesis of benevolence* (HB), which implies that agents always choose the equilibrium desired by the principal;

2) *the principle of maximal guaranteed result* (PMGR), i.e., the principal expects the worst-case equilibrium of the game.

Using either the HB or the PMGR, one has the *problem of informational control* in two settings as follows:

(5) $\max_{x \in \Psi_X(I)} \Phi(x, I) \xrightarrow[I \in \Im]{} \max;$

(6) $\min_{x \in \Psi_X(I)} \Phi(x, I) \xrightarrow[I \in \Im]{} \max.$

Naturally, if for any $I \in \Im$ the set $\Psi_X(I)$ consists of a single element, formulas (1) and (2) coincide.

In the sequel, the problem (5) (alternatively, (6)) is called the *informational control problem in the form of the goal function*.

Now, provide an alternative formulation to the problem of informational control (being independent from the goal function of the principal). Assume that the principal wants agents choose an action vector $x \in X'$. The question arises, "For which vectors and by which awareness structure I would the principal achieve this?" In other words, the second possible formulation of the informational control problem is to find the following components. First, the *attainability set*, *viz*, the one composed of the vectors $x \in X'$ such that for each of them the set of awareness structures $\Psi_I(x) \cap \Im$

(7) is nonempty

or

(8) consists of a single element.

Second, the corresponding feasible *awareness structures* $I \in \Psi_I(x) \cap \Im$, meeting the above property for each vector x. Note that the condition (7) "corresponds" to the HB, while the one of (8) "corresponds" to the PMGR. The problem (7) (alternatively, (8)) will be referred to as *the problem of informational control in the form of the attainability set*. Once again, we underline that the second formulation of the problem does not depend on the goal function of the principal. It merely reflects the possibility of bringing the system to a certain state by informational control.

Methods and examples of problems (5)-(8) solution are described in [11, 62].

Particular case is the *concordant informational control*; here agents are informed about the fact of control implementation by a principal, and still they trust messages of the principal. Evidently, implementing such control requires specific conditions [10].

REFLEXIVE STRUCTURES AND REFLEXIVE EQUILIBRIUM. Publications on strategic reflexion models, the so-called level k models, appeared in the mid-1990s [18, 54, 71]. In 2004, they were generalized by the *cognitive hierarchies model* (CHM) [7].

The survey [78] identified four basic approaches to the construction and study of strategic reflexion models within the framework of game theory and experimental economics. We cite fundamental works only (references to later research can be found in [78]). Notably, the four basic approaches are:

– the level k approach [17];
– the approach of quantal best response equilibria [49];
– the quantal level k approach [70];
– the approach of cognitive hierarchies [7].

All of this approaches are mainly generalized by the following model. The hypothesis of indicator behavior implies that choosing his/her actions by the procedure (2), an agent does not ponder over that the rest agents act similarly. Otherwise, an agent would perform reflexion and (making decisions at a time instant t) seek for the best response to the actions of the rest agents, forecasted according to (2). In this case, the state of goal is no more defined by formula (1). Instead, we obtain

(9) $w_i(x_{-i}^t) = \arg \max_{y \in \Re^1} F_i(y, x_{-i}^t).$

Here x_{-i}^t satisfies (1). We will believe that a reflexing agent of rank 1 considers the rest agents as non-reflexing.

Similarly, it is possible to consider agents with higher reflexion ranks (the term "an agent of reflexion rank k" possesses many synonyms (a step k player, a level k player, a k-level player, a smart k-player, etc – see [47, 69] and the survey [60]). For this, define $\aleph = \{N_0, N_1, \ldots, N_m\}$ as a partition of the agents' set N, where N_i is the set of agents with reflexion rank i, $i = \overline{0, m}$, and m specifies the *maximal reflexion rank*, $n_i = |N_i|$, $i \in N$, $\sum_{i=0}^{m} n_i = n$. We will call $\aleph$ a *reflexive partition* [38].

Suppose that an agent with reflexion rank k exactly knows the sets (shares) of the agents with ranks $k' < k - 1$. Moreover, assume that he/she considers all agents as having reflexion rank $k - 1$. In other words, this agent does not concede the existence of agents with the same (or even higher) reflexion rank than his/her rank. In addition, the agent in question may incorrectly estimate the sets of agents possessing reflexion ranks $k - 1$, k,....

Consider a given initial action vector x^0 of the agents. Let us study the following dynamic reflexive model of their decision making. The corresponding expressions for the one-step "game" model represent a special case, when decisions are made one-time under $\gamma_i^1 \equiv 1, i \in N$.

Reflexion rank 0. Take agents with reflexion rank 0 (belonging to the set N_0). Assume that they choose actions, thinking that the rest agents act similarly to the previous period. Formula (1) yields

(10) $x_i^t = x_i^{t-1} + \gamma_i^t \, [w_i(x_{-i}^{t-1}) - x_i^{t-1}], i \in N_0, t = 1, 2, \ldots$.

In the case $N_0 = N$ (no reflexing agents), all agents observe the *real trajectory* $(x^0, \ldots, x^t, \ldots)$ of the agents' action vectors, see (10).

Reflexion rank 1. Agent j with reflexion rank 1 $(j \in N_1)$ considers the rest agents as having reflexion rank 0. According to formula (10), he/she "forecasts" their choice. Hence, his/her choice $x1_j^t$ represents the best response on the outcome expected by this agent:

(11) $x1_j^t = x1_j^{t-1} + \gamma_j^t \, [w_j(x_{-j}^t) - x1_j^{t-1}], j \in N_1$.

For agent $j \in N_1$, the *forecasted trajectory* is defined by $(x^0, \ldots, (x1_j^t, x_{-j}^t), \ldots)$; however, actually the trajectory $(x^0, \ldots, (x1_{j\in N_1}^t, x_{i\in N_0}^t), \ldots)$ is realized. This means that the real trajectory may differ from the forecasted trajectories of agents with reflexion ranks 0 and 1 [62].

Reflexion rank 2. Suppose that each agent j with reflexion rank 2 $(j \in N_2)$ exactly knows the set N_0; moreover, he/she considers all agents from the set $N_1 \cup N_2 \setminus \{j\}$ as having reflexion rank 1. In the general case of several agents with reflexion rank 2, this agent wrongly assigns rank 1 to them. Consequently, he/she can "forecast" the behavior of the opponents. Therefore, his/her choice is the best response to the expected outcome:

(12) $x2_j^t = x2_j^{t-1} + \gamma_j^t \, [w_j(x_{i\in N_0}^t, \, x1_{l\in N_1 \cup N_2 \setminus \{j\}}^t) - x2_j^{t-1}], j \in N_2$.

For agent $j \in N_2$, the *forecasted trajectory* is given by $(x^0, \ldots, (x2_j^t, \, x1_{l\in N_1\cup N_2\setminus\{j\}}^t, \, x_{i\in N_0}^t), \ldots)$, while actually the trajectory $(x^0, \ldots, (x2_{j\in N_2}^t, x1_{l\in N_1}^t, x_{l\in N_0}^t), \ldots)$ is realized.

Reflexion rank *k* $(k \le m)$. The behavior of agents with reflexion rank k is described by analogy to the three cases above (reflexion ranks 0, 1 and 2). This is done on the basis of the following *awareness structure* of the agents. For agent j with reflexion rank k, denote by $\aleph_{jk}$ the *subjective reflexive partition* (the beliefs of the agent about the partitions of all agents):

(13) $\aleph_{jk} = (\underbrace{N_0, N_1, \ldots, N_{k-2}, N_{k-1} \cup N_k \cup \ldots \cup N_m \setminus \{j\}}_{k}, \{j\}, \underbrace{\varnothing, \ldots, \varnothing}_{m-k-1}), j \in N_k$.

An agent with reflexion rank k chooses actions by the procedure

(14) $xk_j^t = xk_j^{t-1} + \gamma_j^t \, [w_j(x_{l\in N_0}^t, \, x1_{l\in N_1}^t, \, \ldots, \, x[k-1]_{l\in N_{k-1}\cup N_k\cup\ldots\cup N_m/\{j\}}^t) - xk_j^{t-1}], j \in N_k$.

In the "static" case, this agent selects the action

(15) $xk_j^*(\aleph_{jk}) = \arg\max_{y\in\Re^1} F_j(y, \, x_{l\in N_0}^1, \, x1_{l\in N_1}^1, \, \ldots, \, x[k-1]_{l\in N_{k-1}\cup N_k\cup\ldots\cup N_m/\{j\}}^1), j \in N_k$.

Therefore, a ***reflexive structure*** represents the set of subjective reflexive partitions of all agents. Assume that agents' beliefs about the reflexion ranks of each other satisfy (13). Then the awareness structure is uniquely defined by the reflexive partition $\aleph$.

The vector of agents' actions

(16) $x^*(\aleph) = \{ xk_j^*(\aleph_{jk}) \}_{j\in N_k, k=\overline{0,m}}$

is said to be a ***reflexive equilibrium*** of the game $\Gamma_\aleph = \{N, F_i(\cdot)_{i \in N}, \aleph\}$ [38, 60]. In other words, a reflexive equilibrium forms the set of agents' actions being the best responses to opponents' actions (according to an existing reflexive structure). By virtue of the assumptions regarding the existence and uniqueness of best responses, a reflexive equilibrium always exists. Furthermore, a reflexive equilibrium seems rather exotic. Generally, the actions of agents are not the best responses to opponents' actions. Detailed classification of strategic reflexion models is given in [38, 60].

The described general model of reflexive *collective behavior* would hardly lead to general analytical derivations. Nevertheless, it may provide a basis for developing particular analytical models or general simulation models (e.g., according to the classification suggested in [58]). Such models serve for describing and forecasting collective behavior (human beings, mobile robots, program agents) in various situations. For instance, we refer an interested reader to [62] for reflexive simulation models of evacuation, reflexive models of transport flows and other numerous examples from different applications.

By proper variation of reflexive partitions, one can change the actions of agents, i.e., perform ***reflexive control*** [38, 62]. Consider reflexive partition as a control parameter. It is possible to formulate *controllability problem*, as follows. Under a given set $\Im$ of feasible reflexive partitions, find the set of agents' action vectors $X(\Im) = \bigcup_{\aleph \in \Im} x(\aleph)$ that can be realized by reflexive control. The inverse problem lies in obtaining the "minimal" set of feasible reflexive partitions (in a certain sense), allowing to realize a given agents' action vector.

Now, let us address the control problem. Suppose that the preferences of a control subject (a *principal*) are described by his/her real-valued goal function $F_0(Q(x^*))$ defined on the set of *aggregated outcomes* ($Q: \Re^n \to \Re^1$), i.e., $F_0(\cdot): \Re^1 \to \Re^1$. Using the expression (16), the *efficiency of the reflexive partition* $\aleph$ can be characterized by $K(\aleph) = F_0(Q(x^*(\aleph)))$.

Consequently, the *problem of reflexive control* (in terms of reflexive partitions) can be formally stated as [38, 62]

(17) $K(\aleph) \to \max_{\aleph \in \Im}$.

Let K_m be the maximal value of the efficiency criterion in the problem (17) under a fixed maximal reflexion rank m . The problem of the *maximal rational rank of reflexion* (a rank being pointless to exceed for the principal in the sense of controllability or/and efficiency of reflexive control) is to find: $m^* = \min \{m \mid m \in \operatorname{Arg} \max_{w=0,1,2,\ldots} K_w\}$.

To proceed, we discuss conformity of subjective reflexive partitions of the agents. Suppose that each agent observes merely the aggregated outcome. Trajectories forecasted by the agents may differ from the real trajectory (see the general reflexive model of collective behavior). This motivates the agents to doubt the correctness of their subjective reflexive partitions. Imagine that the agents observe just the aggregated outcome of the game (in addition to their own actions). By analogy to the condition of stable informational control (see above), one can introduce the *condition of stable reflexive partition*. Notably, require that the aggregated outcome for the real trajectory coincides with the forecasted aggregated outcomes for all agents. Stability of reflexive partitions is closely associated with learning in games. Observing the behavior of opponents (which differs from the forecasted behavior), agents may modify their beliefs about the reflexion ranks of the opponents or pass to higher levels of reflexion. Under a fixed reflexive partition $\aleph \in \Im$, we have realization of the action vector (16). And the aggregated outcome $Q(x^*(\aleph))$ is realized. According to agent j with reflexion rank k , the following vector is realized:

$$\tilde{x}_{jk}(\aleph_{jk}) = (x_{l \in N_0},\ x1_{l \in N_1},\ x2_{l \in N_2},\ \ldots,\ x[k-1]_{l \in N_{k-1} \cup N_k \cup \ldots \cup N_m \setminus \{j\}},\ xk_j), j \in N_k, k = \overline{0, m} .$$

The condition of stable reflexive partition $\aleph \in \Im$ takes the form $Q(\tilde{x}_{jk}(\aleph_{jk})) = Q(x^*(\aleph)), j \in N_k, k = \overline{0, m}$. The problem of reflexive control ($\aleph$) can be stated on the set of stable reflexive controls (if nonempty). In practice, this means that the principal forms an optimal partition of the agents into reflexion ranks. In such partition, the agents do not doubt the correctness of their beliefs about reflexion ranks of the opponents (based on observing the results of the "game").

CONCLUSION. Mathematical models of informational and/or reflexive structures and equilibria (and models of corresponding control problems) allow the following:

• from the decision theory viewpoint, extending the class of collective behavior models for intelligent agents performing a joint activity under incomplete awareness and missed common knowledge;

• from the descriptive viewpoint, enlarging the set of outcomes that can be "explained" (within the framework of the model) as stable results of agents' interaction; accordingly, extending the controllability domain (for control problems);

• from the normative viewpoint, posing/solving the problems of collective behavior by choosing a proper structure of agents' awareness.

Numerous applied models of informational or/and reflexive control in economic, social and organizational systems, military problems and other fields are described in [11, 62, 64].

As strategic objectives of future investigations, we mention integration of informational reflexion models with strategic reflexion ones. In other words, it seems promising to construct a language for uniform joint description of informational and reflexive structures.

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